MANIN'S AND PEYRE'S CONJECTURES ON RATIONAL POINTS AND ADELIC MIXING

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Dedicated to Prof. Gregory Margulis on the occasion of his sixtieth birthday

Abstract. Let X be the wonderful compactification of a connected adjoint semisimple group G defined over a number field K. We prove Manin's conjecture on the asymptotic (as $T \to \infty$) of the number of K-rational points of X of height less than T, and give an explicit construction of a measure on $X(\mathbb{A})$, generalizing Peyre's measure, which describes the asymptotic distribution of the rational points $\mathbf{G}(K)$ on $X(\mathbb{A})$. Our approach is based on the mixing property of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ which we obtain with a rate of convergence.

Soit X la compactification merveilleuse d'un groupe semisimple $\mathbf{G}$, connexe, de type adjoint, algébrique défini sur un corps de nombre K. Nous démontrons l'asymptotique conjecturée par Manin du nombre de points K-rationnels sur X de hauteur plus petite que T, lorsque $T \to +\infty$, et construisons de manière explicite une mesure sur $X(\mathbb{A})$, généralisant celle de Peyre, qui décrit la répartition asymptotique des points rationnels $\mathbf{G}(K)$ sur $X(\mathbb{A})$. Ce travail repose sur la propriété de mélange de $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, qui est démontrée avec une estimée de vitesse.

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1. Introduction

Let K be a number field and X a smooth projective variety defined over K. A fundamental problem in modern algebro-arithmetic geometry is to describe the set $X(K)$ in terms of the geometric invariants of X. One of the main conjectures in this area was made by Manin in the late eighties in [1]. It formulates the asymptotic (as $T \to \infty$) of the number of points in $X(K)$ of height less than T for Fano varieties (that is, varieties with ample anti-canonical class).

Manin's conjecture has been proved for flag varieties ([27], [42]), toric varieties ([2], [3]), horospherical varieties [53], equivariant compactifications of unipotent groups (see [14], [47], [48]), etc. We refer to survey papers by Tschinkel ([54], [55]) for a more precise background on this conjecture. Recently Shalika, Tschinkel and Takloo-Bighash proved the conjecture for the wonderful compactification of a connected semisimple adjoint group [50]. In this paper, we present a different proof of the conjecture, as well as describe the asymptotic distributions of rational points of bounded height as conjectured by Peyre. Our proof relies on the computation of the volume asymptotics of height balls in [50]. We refer to [6] for the comparison of these two approaches.

Although our work is highly motivated by conjectures in arithmetic geometry, our approach is almost purely (algebraic) group theoretic. For this reason, we formulate our main results in the language of algebraic groups and their representations in the introduction and refer to section 7 for the account that how these results imply the conjectures of Manin and Peyre.

1.1. Height function.

We begin by defining the notion of a height function on the K-rational points of the projective n-space $\mathbb{P}^n$. Intuitively speaking, the height of a rational point $x \in \mathbb{P}^n(K)$ measures an *arithmetic* size of x. In the case of $K = \mathbb{Q}$, it is simply given by

$$\mathrm{H}(x) = \max_{0 \le i \le n} |x_i|$$

where $(x_0, \cdots, x_n)$ is a primitive integral vector representing x. To give its definition for a general K, we denote by R the set of all normalized absolute values $x \mapsto |x|_v$ of K, and by K_v the completion of K with respect to $|\cdot|_v$. For each $v \in R$, choose a norm H_v on K_v^{n+1} which is simply the max norm $\mathrm{H}_v(x_0, \cdots, x_n) = \max_{i=0}^n |x_i|_v$ for almost all v. Then a function $\mathrm{H} : \mathbb{P}^n(K) \to \mathbb{R}_{>0}$ of the following form is called a height function:

$$\mathrm{H}(x) := \prod_{v \in R} \mathrm{H}_v(x_0, \cdots, x_n)$$

for $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(K)$. Since $\mathrm{H}_v(x_0, \cdots, x_n) = 1$ for almost all $v \in R$, we have $\mathrm{H}(x) > 0$ and by the product formula, H is well defined, i.e., independent of the choice of representative for x.

It is easy to see that for any $T > 0$, the number

$$N(T) := \#\{x \in \mathbb{P}^n(K) : \mathrm{H}(x) < T\}$$

is finite. Schanuel [46] computed the precise asymptotic in 1964:

$$N(T) \sim c \cdot T^{n+1} \quad \text{as } T \to \infty$$

for some explicit constant $c = c(\mathrm{H}) > 0$.

Unless mentioned otherwise, throughout the introduction, we let $\mathbf{G}$ be a connected semisimple adjoint group over K and $\iota : \mathbf{G} \to \mathrm{GL}_N$ be a faithful representation of $\mathbf{G}$ defined over K with a unique maximal weight. Consider the projective embedding of $\mathbf{G}$ over K induced by ι:

$$\bar{\iota} : \mathbf{G} \to \mathbb{P}(\mathrm{M}_N)$$

where M_N denotes the space of matrices of order N. We then define a height function H_ι on $\mathbf{G}(K)$ associated to ι by pulling back a height function on $\mathbb{P}(\mathrm{M}_N(K))$ via $\bar{\iota}$. That is, for $g \in \mathbf{G}(K)$,

$$\mathrm{H}_\iota(g) := \prod_{v \in R} \mathrm{H}_v(\iota(g)), \tag{1.1}$$

where each H_v is a norm on $\mathrm{M}_N(K_v)$, which is the max norm for almost all $v \in R$.

We note that H_ι is not uniquely determined by ι, because of the freedom of choosing H_v locally (though only for finitely many v).

1.2. **Asymptotic number of rational points.** For each $T > 0$, we introduce the notation for the number of points in $\mathbf{G}(K)$ of height less than T:

$$N(\mathrm{H}_\iota, T) := \#\{g \in \mathbf{G}(K) \ : \ \mathrm{H}_\iota(g) \ < \ T\}.$$

Theorem 1.2. *There exist $a_\iota \in \mathbb{Q}^+$, $b_\iota \in \mathbb{N}$ and $c = c(\mathrm{H}_\iota) > 0$ such that for some $\delta > 0$,*

$$N(\mathrm{H}_\iota, T) = c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \cdot (1 + O((\log T)^{-\delta})).$$

The constants a_ι and b_ι can be defined explicitly by combinatorial data on the root system of $\mathbf{G}$ and the unique maximal weight of ι. Choose a maximal torus $\mathbf{T}$ of $\mathbf{G}$ defined over K containing a maximal K-split torus and a set Δ of simple roots in the root system $\Phi(\mathbf{G}, \mathbf{T})$. Denote by 2ρ the sum of all positive roots in $\Phi(\mathbf{G}, \mathbf{T})$, and by λ_ι the maximal weight of ι. Define $u_\alpha, m_\alpha \in \mathbb{N}$, $\alpha \in \Delta$, by

$$2\rho = \sum_{\alpha \in \Delta} u_\alpha \alpha \quad \text{and} \quad \lambda_\iota = \sum_{\alpha \in \Delta} m_\alpha \alpha.$$

The fact that $m_\alpha \in \mathbb{N}$ follows since $\mathbf{G}$ is of adjoint type. Consider the twisted action of the Galois group $\Gamma_K := \mathrm{Gal}(\bar{K}/K)$ on Δ (for instance, if the K-form of $\mathbf{G}$ is inner, this action is just trivial). Then

$$a_\iota = \max_{\alpha \in \Delta} \frac{u_\alpha + 1}{m_\alpha} \quad \text{and} \quad b_\iota = \#\{\Gamma_K.\alpha : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}. \tag{1.3}$$

Note that the exponent a_ι is independent of the field K, and b_ι depends only on the quasi-split K-form of $\mathbf{G}$. Therefore, by passing to a finite field extension containing the splitting field of $\mathbf{G}$, b_ι also becomes independent of K.

Remark: When $\mathbf{G}$ is almost K-simple or, more generally, when H_ι is the product of height functions of the K-simple factors of $\mathbf{G}$, we can improve the rate of convergence in Theorem 1.2: for some $\delta > 0$,

$$N(\mathrm{H}_\iota, T) = c \cdot T^{a_\iota} P(\log T) \cdot (1 + O(T^{-\delta}))$$

where $P(x)$ is a monic polynomial of degree $b_\iota - 1$.

1.3. **Distribution of rational points.** For each $v \in R$, denote by $X_{\iota,v}$ the closure of $\bar{\iota}(\mathbf{G}(K_v))$ in $\mathbb{P}(\mathrm{M}_N(K_v))$, and consider the compact space $X_\iota := \prod_{v\in R} X_{\iota,v}$. In section 6, we construct a probability measure μ_ι on X_ι which describes the asymptotic distribution of rational points in $\mathbf{G}(K)$ in X_ι with respect to the height H_ι. To keep the introduction concise, we give the definition of μ_ι only when ι is saturated. A representation $\iota : \mathbf{G} \to \mathrm{GL}_N$ is called *saturated* if the set

$$\{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}$$

is not contained in the root system of a proper normal K-subgroup of $\mathbf{G}$. In particular, if $\mathbf{G}$ is almost K-simple, any representation of $\mathbf{G}$ is saturated.

Let τ denote the Haar measure on $\mathbf{G}(\mathbb{A})$ such that $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$. Denote by Λ the set of all automorphic characters of $\mathbf{G}(\mathbb{A})$ (cf. section 2.4) and by W_ι the maximal compact subgroup of the group $\mathbf{G}(\mathbb{A}_f)$ of finite adeles, under which H_ι is bi-invariant (see Definition 2.7). Then the following is a positive real number (see Propositions 4.6 and 4.11 (3), noting $r_\iota = \gamma_{W_\iota}(e)$ in the notation therein):

$$r_\iota := \sum_{\chi\in\Lambda} \lim_{s\to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s} \chi(g)\, d\tau(g). \tag{1.4}$$

For ι saturated, the probability measure μ_ι on X_ι is the unique measure satisfying that for any $\psi \in C(X_\iota)$ invariant under a co-finite subgroup of W_ι,

$$\mu_\iota(\psi) = r_\iota^{-1} \cdot \sum_{\chi\in\Lambda} \lim_{s\to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s} \chi(g)\, \psi(g)\, d\tau(g) \tag{1.5}$$

(see Theorem 4.18). We refer to (6.16) for the definition of μ_ι for a general ι:

Theorem 1.6. *For any $\psi \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{N(\mathrm{H}_\iota, T)} \sum_{g\in\mathbf{G}(K):\mathrm{H}_\iota(g)<T} \psi(g) = \int_{X_\iota} \psi\, d\mu_\iota.$$

Remark 1.7. (1) For ι saturated, the measure μ_ι coincides with the measure $\tilde{\mu}_\iota$ which describes the distribution of height balls in $\mathbf{G}(\mathbb{A})$ (see Proposition 4.27).

(2) Although the projection $\mu_{\iota,S}$ of μ_ι to $X_{\iota,S} = \prod_{v\in S} X_{\iota,v}$ is always equivalent to a Haar measure on $\mathbf{G}_S = \prod_{v\in S} \mathbf{G}(K_v)$ (Proposition 4.22), it is $\mathbf{G}_S$-invariant, only when the height $\mathrm{H}_{\iota,S} = \prod_{v\in S} \mathrm{H}_v \circ \iota$ is $\mathbf{G}_S$-invariant.

(3) The space $X_{\iota,S}$ is a compactification of $\mathbf{G}_S$ which is an analog of the Satake compactification defined for real groups (see, for example, [9]). Theorem 1.6 implies that the rational points $\mathbf{G}(K)$ do not escape to the boundary $X_{\iota,S} - \mathbf{G}_S$. It is interesting to compare this result with the distribution of the integral points $\mathbf{G}(\mathbb{Z})$ of bounded height in the Satake compactification of $\mathbf{G}(\mathbb{R})$ where the limiting distribution is supported on the boundary (see [30] and [39] for more details).

1.4. **Counting and volume heuristic.** To explain our strategy in counting K-rational points of $\mathbf{G}$, we first recall the analogous results in counting integral points in a simple real algebraic group. Let $G \subset \mathrm{GL}_N$ be a connected non-compact simple real algebraic group and Γ be a lattice in G, i.e., a discrete subgroup of finite co-volume. Fixing a norm $\|\cdot\|$ on $\mathrm{M}_N(\mathbb{R})$, set $B_T := \{g \in G : \|g\| \le T\}$. By Duke-Rudnick-Sarnak [23] and Eskin-McMullen [24] independently, it is well known that

$$\#\Gamma \cap B_T \sim \int_{B_T} dg \quad \text{as } T \to \infty, \tag{1.8}$$

where dg is the Haar measure on G such that $\int_{\Gamma\backslash G} dg = 1$.

Coming back to the question of counting rational points $\mathbf{G}(K)$, we recall that $\mathbf{G}(K)$ is a lattice in the adele group $\mathbf{G}(\mathbb{A})$ when embedded diagonally and that the height function $\mathrm{H}_\iota = \prod_{v\in R} \mathrm{H}_v \circ \iota$ on $\mathbf{G}(K)$ extends to $\mathbf{G}(\mathbb{A})$.

If we set

$$B_T := \{g \in \mathbf{G}(\mathbb{A}) : \mathrm{H}_\iota(g) < T\},$$

then B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$ (Lemma 2.5) and we have the equality

$$N(\mathrm{H}_\iota, T) = \#\mathbf{G}(K) \cap B_T.$$

In view of (1.8), one naturally asks whether the following holds:

$$\#\mathbf{G}(K) \cap B_T \sim \tau(B_T) \quad \text{as } T \to \infty. \tag{1.9}$$

It turns out that the group $\mathbf{G}(\mathbb{A})$ is too big for (1.9) to hold in general, due to the presence of non-trivial automorphic characters of $\mathbf{G}(\mathbb{A})$. For a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$, denote by $\Lambda^{W_f} \subset \Lambda$ the set of all W_f-invariant characters in Λ. We set

$$G_{W_f} := \ker(\Lambda^{W_f}) = \cap\{\ker \chi \subset \mathbf{G}(\mathbb{A}) : \chi \in \Lambda^{W_f}\}.$$

The subgroup G_{W_f} is a normal subgroup of $\mathbf{G}(\mathbb{A})$ with finite index (see Lemma 4.7), and hence $\mathbf{G}(K)$ is a lattice in G_{W_f}. Denote by τ_{W_f} the Haar measure on G_{W_f} normalized so that $\tau_{W_f}(\mathbf{G}(K)\backslash G_{W_f}) = 1$.

Theorem 1.10. *Assume that $\iota : \mathbf{G} \to \mathrm{GL}_N$ is saturated. Then for any compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant,*

$$\#\mathbf{G}(K) \cap B_T \sim_T \tau_{W_f}(G_{W_f} \cap B_T).$$

We remark that one cannot in general replace G_{W_f} by $\mathbf{G}(\mathbb{A})$ (see example 4.24), and Theorem 1.10 does not hold for ι non-saturated.

As in the proof of Eskin-McMullen of (1.8), our key ingredient in proving Theorem 1.10 is the mixing theorem on $L^2(\mathbf{G}(K)\backslash G_{W_f})$.

1.5. **Adelic mixing.** Let $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ denote the orthogonal complement in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ to the direct sum of all automorphic characters. In the case when $\mathbf{G}$ is simply connected, $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ coincides with the orthogonal complement $L^2_0(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ to the constant functions.

Set $\mathbf{G}_\infty := \prod_{v\in R_\infty} \mathbf{G}(K_v)$ where R_∞ is the subset of R of all archimedean valuations.

Theorem 1.11 (Automorphic bound for $\mathbf{G}$)**.** *Let $\mathbf{G}$ be a connected absolutely almost simple K-group. Let U_∞ be a maximal compact subgroup of $\mathbf{G}_\infty$ and W_f a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then there exist $c_{W_f} > 0$ and $r_0 = r_0(\mathbf{G}_\infty) > 0$ such that for any U_∞-finite and W_f-invariant functions $\psi_1, \psi_2 \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$|\langle\psi_1, g.\psi_2\rangle| \le c_{W_f}\cdot(\dim\langle U_\infty\psi_1\rangle\cdot\dim\langle U_\infty\psi_2\rangle)^{r_0}\cdot\tilde{\xi}_{\mathbf{G}}(g)\cdot\|\psi_1\|_2\cdot\|\psi_2\|_2 \quad \text{for all } g \in \mathbf{G}(\mathbb{A}).$$

Here, $\tilde{\xi}_{\mathbf{G}} : \mathbf{G}(\mathbb{A}) \to (0, 1]$ is an explicitly constructed proper function which is L^p-integrable for some $p = p(\mathbf{G}) < \infty$. (see Def. 3.18).

Using the restriction of scalars functor, we extend this theorem to connected almost K-simple adjoint (simply connected) groups (Theorem 3.20). The above bounds on matrix coefficients can also be extended to smooth functions in certain Sobolev spaces (see Theorem 3.25). We also mention a paper of Guilloux [31] where an application of Theorem 1.11 was discussed in local-global principle problems.

Corollary 1.12 (Adelic Mixing)**.** *Let $\mathbf{G}$ be a connected absolutely simple K-group, or a connected almost K-simple adjoint (simply connected) K-group. Then for any $\psi_1, \psi_2 \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$\langle\psi_1, g.\psi_2\rangle \to 0$$

as $g \in \mathbf{G}(\mathbb{A})$ tends to infinity.

Any W_f-invariant function in $L^2(\mathbf{G}(K)\backslash G_{W_f})$ orthogonal to constants belongs to $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ (see Lemma 4.12). Hence Corollary 1.12 implies that if ψ_1 and ψ_2 are W_f-invariant functions in $L^2(\mathbf{G}(K)\backslash G_{W_f})$, then as $g \to \infty$,

$$\int_{\mathbf{G}(K)\backslash G_{W_f}} \psi_1(x)\psi_2(xg)\, d\tau_{W_f}(x) \to \int \psi_1\, d\tau_{W_f} \cdot \int \psi_2\, d\tau_{W_f}.$$

For each $v \in R$, denote $\hat{\mathbf{G}}_v^{\mathrm{Aut}} \subset \hat{\mathbf{G}}_v$ the automorphic dual of $\mathbf{G}(K_v)$, i.e., the subset of unitary dual of $\mathbf{G}(K_v)$ consisting of representations which are weakly contained in the representations appearing as $\mathbf{G}(K_v)$ components of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^{O_f}$ for some compact open subgroup O_f of $\mathbf{G}(\mathbb{A}_f)$. The proof of Theorem 1.11 goes roughly as follows: if $\tilde{\xi}_v$ is a uniform bound for the matrix coefficients of infinite dimensional representations in $\hat{\mathbf{G}}_v^{\mathrm{Aut}}$, $\tilde{\xi}_{\mathbf{G}}$ is defined to be the product $\prod_{v\in R}\tilde{\xi}_v$. This can be made precise using the language of direct integral of a representation (cf. proof of Theorem 3.10). For those $v \in R$ such that the K_v-rank of $\mathbf{G}$ is at least 2, the uniform bounds, say ξ_v, of matrix coefficients of *all* infinite dimensional unitary representations of $\mathbf{G}(K_v)$ were obtained by Oh [40]. For these cases, one can simply take $\tilde{\xi}_v = \xi_v$. In particular, if K-rank of $\mathbf{G}$ is at least 2 and $\mathbf{G}(K_v)^+$ denotes the closed subgroup of $\mathbf{G}(K_v)$ generated by all unipotent elements in $\mathbf{G}(K_v)$, we have $\tilde{\xi}_{\mathbf{G}} = \prod_{v\in R}\xi_v$ and $\tilde{\xi}_{\mathbf{G}}$ works as a uniform bound for all unitary representations of $\mathbf{G}(\mathbb{A})$ without $\mathbf{G}(K_v)^+$-invariant vectors for each $v \in R$ (see Theorem 3.10 for a precise statement). Moreover $\tilde{\xi}_{\mathbf{G}}$ is fairly sharp in these cases. For instance, one can show that $\tilde{\xi}_{\mathbf{G}}$ is optimal for $\mathbf{G} = \mathrm{SL}_n$ $(n \geq 3)$, or Sp_{2n} $(n \geq 2)$ by [17, 5.4].

When there is $v \in R$ with K_v-rank of $\mathbf{G}$ one, finding an automorphic bound $\tilde{\xi}_v$ is essentially carried out by Clozel [15]. In particular, several deep theorems in automorphic theory were used such as the Gelbart-Jacquet bound [29] toward Ramanujan conjecture, the results of Burger-Sarnak [12] and Clozel-Ullmo [18] on lifting automorphic bounds, the base changes by Rogawski [45] and Clozel [16], and Jacquet-Langlands correspondence [34].

1.6. **Organization of the paper.** In section 2 we list some notations and preliminaries which will be used throughout the paper. In section 3, we discuss adelic mixing and prove Theorem 1.11. We also extend a theorem of Clozel-Oh-Ullmo on the equidistribution of Hecke points [17] in this section as an application of adelic mixing. In section 4, we deduce the volume asymptotics of height balls from the results in [50] and construct in subsection 4.4 the probability measure $\tilde{\mu}_\iota$ on X_ι which describes the asymptotic distribution of height balls in $\mathbf{G}(\mathbb{A})$. The main theorem in section 5 is Theorem 5.2 on the equidistribution of rational points with respect to $\tilde{\mu}_\iota$ for the case when ι is saturated. Theorems 1.6 (saturated cases) and 1.10 follow from this theorem. In section 6, we prove Theorem 1.6 for general cases (Theorem 6.2) as well as Theorem 1.2 (Theorem 6.17). In section 7, we restate our main theorems in the context of Manin's and Peyre's conjectures.

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2. Notations and Preliminaries

We set up some notations which will be used throughout the paper. Let K be a number field and $\mathbf{G}$ a connected semisimple group defined over K. We denote by R_K, or simply by R, the set of all normalized absolute values on K. We keep the same notation R, R_f, K_v as in the introduction.

2.1. Let $\mathcal{O}$ denote the ring of integers of K and $\mathcal{O}_v$ the valuation ring of K_v. Set $R_\infty = R - R_f$. For $v \in R_f$, let q_v denote the order of the residue field of $\mathcal{O}_v$. We choose an absolute value $|\cdot|_v$ on K_v normalized so that the absolute value of a uniformizer of $\mathcal{O}_v$ is given by q_v^{-1}. Denote by $\mathbb{A}$ the adele ring over K and by $\mathbf{G}(\mathbb{A})$ the adele group associated to $\mathbf{G}$.

Denote by $\mathbf{G}(\mathbb{A}_f)$ (resp. $\mathbf{G}_\infty$) the subgroup of finite (resp. infinite) adeles, i.e., $((g_v)_v) \in \mathbf{G}(\mathbb{A})$ with $g_v = e$ for all $v \in R_\infty$ (resp. for all $v \in R_f$). Then

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}_\infty \times \mathbf{G}(\mathbb{A}_f).$$

2.2. We fix a smooth model $\mathcal{G}$ of $\mathbf{G}$ over $\mathcal{O}[k^{-1}]$ for some non-zero $k \in \mathbb{Z}$. There exists a finite subset $S_0 \subset R_f$ such that for any $v \in R_f - S_0$, $\mathbf{G}$ is unramified over K_v and $\mathcal{G}(\mathcal{O}_v)$ is a hyperspecial compact subgroup (cf. [57]). We set $U_v = \mathcal{G}(\mathcal{O}_v)$ for each $v \in R_f - S_0$. Then for each $v \in R_f - S_0$, one has the group A_v of K_v-rational points of a maximal K_v-split torus of $\mathbf{G}$ so that the following Cartan decomposition holds:

$$\mathbf{G}(K_v) = U_v A_v^+ U_v \tag{2.1}$$

where A_v^+ is a closed positive Weyl chamber of A_v. More precisely, one can choose a system Φ_v^+ of positive roots in the set $\Phi_v = \Phi(\mathbf{G}(K_v), A_v)$ of all non-multipliable roots of $\mathbf{G}(K_v)$ relative to A_v so that

$$A_v^+ = \{a \in A_v : \alpha(a) \geq 1 \text{ for each } \alpha \in \Phi_v^+\} \quad \text{for } v \text{ archimedean}$$

$$A_v^+ = \{a \in A_v : |\alpha(a)|_v \in q_v^{\mathbb{N}} \text{ for each } \alpha \in \Phi_v^+\} \quad \text{otherwise.}$$

For $v \in S_0 \cup R_\infty$, there exists a good maximal compact subgroup U_v (cf. [40, 2.1] for definition) of $\mathbf{G}(K_v)$ such that

$$\mathbf{G}(K_v) = U_v A_v^+ \Omega_v U_v$$

where Ω_v is a finite subset in the centralizer of A_v in $\mathbf{G}(K_v)$.

In particular for any $g \in \mathbf{G}(K_v)$, there exist unique $a_v \in A_v^+$ and $d_v \in \Omega_v$ such that $g \in U_v a_v d_v U_v$. For $v \in R_\infty$, any maximal compact subgroup of $\mathbf{G}(K_v)$ is a good maximal compact subgroup and $\Omega_v = \{e\}$.

2.3. For a finite subset S of R, let $\mathbf{G}^S$ denote the subgroup of $\mathbf{G}(\mathbb{A})$ consisting of (g_v), with $g_v = e$ for all $v \in S$, and set $\mathbf{G}_S := \prod_{v\in S} \mathbf{G}(K_v)$. Note that $\mathbf{G}(\mathbb{A}) = \mathbf{G}_S\mathbf{G}^S$. For each $v \in R$, let τ_v, or dg_v, denote a Haar measure on $\mathbf{G}(K_v)$ such that $\tau_v(U_v) = 1$ whenever $v \in R_f$. Then the collection $\{\tau_v : v \in R\}$ defines a Haar measure, say τ, on $\mathbf{G}(\mathbb{A})$ (cf. [44, 3.5]). We will assume that $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$. This is possible by replacing τ_v, $v \in R_\infty$ with a suitable multiple of it, since $\mathbf{G}(K)$ is a lattice in $\mathbf{G}(\mathbb{A})$ (cf. [44, Theorem 5.5]).

We denote by τ_S the product measure $\prod_{v\in S} \tau_v$ on $\mathbf{G}_S$ and by τ^S the Haar measure on $\mathbf{G}^S$ for which the triple (τ, τ_S, τ^S) are compatible with each other, i.e., $\tau = \tau_S \times \tau^S$ locally.

2.4. An automorphic character of $\mathbf{G}(\mathbb{A})$ is a continuous homomorphism from $\mathbf{G}(\mathbb{A})$ to the unit circle $\{z \in \mathbb{C} : z\bar{z} = 1\}$ which contains $\mathbf{G}(K)$ in its kernel. Each automorphic character χ of $\mathbf{G}(\mathbb{A})$ can be considered as a function on the quotient $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$, and since $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$, χ belongs to $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ and $\|\chi\|_2 = 1$. Let Λ denote the set of all automorphic characters of $\mathbf{G}(\mathbb{A})$. Note that any two distinct elements of Λ are orthogonal to each other. We then have an orthogonal decomposition

$$L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) \oplus \hat{\bigoplus}_{\chi\in\Lambda} \mathbb{C}\chi.$$

where $\hat{\bigoplus}_{\chi\in\Lambda} \mathbb{C}\chi$ is the closure of the direct sum of $\mathbb{C}\chi$'s, and $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ denotes its orthogonal complement in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$.

If $\mathbf{G}$ is simply connected, it follows from the strong approximation property that the only automorphic character of $\mathbf{G}(\mathbb{A})$ is the trivial one and hence that $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ is the orthogonal complement to the space of constant functions (cf. Lemma 3.22).

2.5. For any compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$, we denote by $\Lambda^{W_f} \subset \Lambda$ the set of all W_f-invariant characters in Λ, i.e., $\Lambda^{W_f} = \{\chi \in \Lambda : \chi(w) = 1 \quad \text{for all } w \in W_f\}$. We set

$$G_{W_f} := \ker(\Lambda^{W_f}) = \cap\{\ker \chi \subset \mathbf{G}(\mathbb{A}) : \chi \in \Lambda^{W_f}\}. \tag{2.2}$$

The subgroup G_{W_f} is a normal subgroup of $\mathbf{G}(\mathbb{A})$ with finite index (see Lemma 4.7), and hence $\mathbf{G}(K)$ is a lattice in G_{W_f}. Denote by τ_{W_f} the Haar measure on G_{W_f} normalized so that $\tau_{W_f}(\mathbf{G}(K)\backslash G_{W_f}) = 1$.

2.6. For a group $\mathbf{G}$ of adjoint type, let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be a faithful representation defined over K. We give a definition of a height function H_ι on $\mathbf{G}(\mathbb{A})$ associated to ι which is slightly more general than those considered in the introduction. It is this class of the functions for which we prove our main theorems.

Definition 2.3. *A height function* $\mathrm{H}_\iota : \mathbf{G}(\mathbb{A}) \to \mathbb{R}^+$ *is defined by the product* $\prod_{v\in R} \mathrm{H}_{\iota,v}$ *where* $\mathrm{H}_{\iota,v}$ *is a function on* $\mathbf{G}(K_v)$ *for* $v \in R$ *satisfying the following:*

(1) *there exists a finite subset* $S \subset R$ *such that*

$$\mathrm{H}_{\iota,v}(g) = \max_{ij} |\iota(g)_{ij}|_v \quad \textit{for all } v \in R - S;$$

(2) *for* $v \in S$, *there exists* $C > 0$ *such that*

$$C^{-1} \cdot \max_{ij} |\iota(g)_{ij}|_v \le \mathrm{H}_{\iota,v}(g) \le C \cdot \max_{ij} |\iota(g)_{ij}|_v;$$

(3) *for any* $v \in S \cap R_\infty$, *there exists* $b > 0$ *such that for any small* $\epsilon > 0$,

$$(1 - b \cdot \epsilon)\, \mathrm{H}_{\iota,v}(x) \le \mathrm{H}_{\iota,v}(gxh) \le (1 + b \cdot \epsilon)\, \mathrm{H}_{\iota,v}(x)$$

for any $x \in \mathbf{G}(K_v)$ *and any* g, h *in the* ϵ*-neighborhood of* e *in* $\mathbf{G}(K_v)$ *with respect to a Riemannian metric;*

(4) *for any* $v \in S \cap R_f$, $\mathrm{H}_{\iota,v}$ *is bi-invariant under a compact open subgroup of* $\mathbf{G}(K_v)$.

Note by (1) that for $(g_v) \in \mathbf{G}(\mathbb{A})$, since $g_v \in \mathcal{G}(\mathcal{O}_v)$ for almost all $v \in R_f$, $\mathrm{H}_{\iota,v}(\iota(g_v)) = 1$ for almost all v, and hence the product $\prod_{v \in R} \mathrm{H}_{\iota,v}(g_v)$ converges.

Note also that the class of height functions defined above does not depend on the choice of a basis of K^N.

We will need the following observation on heights:

Lemma 2.4. *Suppose that* ι *has a unique maximal weight. Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be connected normal algebraic* K*-subgroups of* $\mathbf{G}$ *with* $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2$ *and* $\mathbf{G}_1 \cap \mathbf{G}_2 = \{e\}$. *There exists* $\kappa > 1$ *such that for any* $g_1 \in \mathbf{G}_1(\mathbb{A})$ *and* $g_2 \in \mathbf{G}_2(\mathbb{A})$,

$$\kappa^{-1} \cdot \mathrm{H}_\iota(g_1)\mathrm{H}_\iota(g_2) \le \mathrm{H}_\iota(g_1 g_2) \le \kappa \cdot \mathrm{H}_\iota(g_1)\mathrm{H}_\iota(g_2).$$

Proof. Let λ_ι denote the highest weight of ι. Then there exists a finite subset $S \subset R$ such that for any $v \in R - S$,

$$\mathbf{G}(K_v) = U_v A_v^+ U_v \quad \text{and} \quad \mathrm{H}_v(\iota(g)) = |\lambda_\iota(a)|_v \quad \text{for } g = u_1 a u_2 \in \mathbf{G}(K_v)$$

where U_v and A_v^+ are defined as in (2.2). In particular, it follows that for each $v \in R - S$, and for any $g_1 \in \mathbf{G}_1(K_v)$ and $g_2 \in \mathbf{G}_2(K_v)$,

$$\mathrm{H}_v(\iota(g_1 g_2)) = \mathrm{H}_v(\iota(g_1))\mathrm{H}_v(\iota(g_2)).$$

On the other hand, for $v \in S$, $\mathrm{H}_{\iota,v}$ is equivalent to λ_ι in the sense that there exists $\kappa_v > 1$ such that

$$\kappa_v^{-1} \cdot |\lambda_\iota(a)|_v \le \mathrm{H}_{\iota,v}(g) \le \kappa_v \cdot |\lambda_\iota(a)|_v \quad \text{for } g = u_1 a d u_2 \in U_v A_v^+ \Omega_v U_v = \mathbf{G}(K_v).$$

This implies the lemma. □

For $T > 0$, set

$$B_T := \{g \in \mathbf{G}(\mathbb{A}) : \mathrm{H}_\iota(g) < T\}.$$

Lemma 2.5. (1) *We have*

$$\delta_0 := \inf_{g \in \mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g) > 0. \tag{2.6}$$

(2) *For each $T > 0$, B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$. In other words, the height function $\mathrm{H}_\iota : \mathbf{G}(\mathbb{A}) \to [\delta_0, \infty)$ is proper.*

Proof. By Definition 2.6, there exists a finite subset S such that for all $v \in R - S$, $\mathrm{H}_v(\iota(g)) \geq 1$ for any $g \in \mathbf{G}(K_v)$. Let $0 < \delta \leq 1$ be such that $\mathrm{H}_v(\iota(g)) \geq \delta$ for $v \in S$ and $\delta_1 = \delta^{\#S}$. Then $\mathrm{H}_\iota(g) \geq \delta_1$ for all $g \in \mathbf{G}(\mathbb{A})$. Hence $\delta_0 \geq \delta_1 > 0$.

Note that

$$B_T \subset \mathbf{G}(\mathbb{A}) \cap \prod_v \{g_v \in \mathbf{G}(K_v) : \mathrm{H}_v(\iota(g_v)) \leq \delta^{-1} T\}.$$

Since for almost all $v \in R_f$, $\mathrm{H}_v(\iota(g_v)) \geq q_v$ whenever $g_v \notin \mathcal{G}(\mathcal{O}_v)$, it follows that for some finite subset $S_1 \subset R$, we have

$$B_T \subset \{(g_v)_v \in \mathbf{G}(\mathbb{A}) : \mathrm{H}_v(\iota(g_v)) \leq \delta_0^{-1} T \text{ for } v \in S_1, \;\; g_v \in \mathcal{G}(\mathcal{O}_v) \text{ otherwise}\}.$$

Since the set $\{g_v \in \mathbf{G}(K_v) : \mathrm{H}_v(\iota(g_v)) \leq b\}$ is compact for any $b > 0$, it follows that B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$. □

Definition 2.7. *For a height function H_ι of $\mathbf{G}(\mathbb{A})$, define W_{H_ι}, or simply W_ι, to be*

$$W_\iota = \{w \in \mathbf{G}(\mathbb{A}_f) : \mathrm{H}_\iota(wg) = \mathrm{H}_\iota(gw) = \mathrm{H}_\iota(g) \quad \textit{for all } g \in \mathbf{G}(\mathbb{A})\}.$$

It is easy to check that W_ι is a subgroup of $\mathbf{G}(\mathbb{A}_f)$, and is compact by the above lemma. Hence W_ι is the maximal compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant.

3. Adelic Mixing

3.1. **Definition and properties of $\xi_{\mathbf{G}}$.** Let $\mathbf{G}$ be a connected semisimple algebraic group defined over a number field K. Let $\mathcal{T}$ denote the set of $v \in R$ such that $\mathbf{G}(K_v)$ is compact, that is, $U_v = \mathbf{G}(K_v)$. It is well known that $\mathcal{T}$ is a finite set.

Denote by Φ_v^+ the system of positive roots in the set of all non-multipliable roots of $\mathbf{G}(K_v)$ relative to A_v^+ and choose a maximal strongly orthogonal system $\mathcal{S}_v$ in Φ_v^+ in the sense of [40] (where an explicit construction is also given). For $v \in R - \mathcal{T}$ and $K_v \neq \mathbb{C}$, define the bi-U_v-invariant function $\xi_v = \xi_{\mathbf{G}(K_v)}$ on $\mathbf{G}(K_v)$ (cf. [40]): for each $g = kadk' \in U_v A_v^+ \Omega_v U_v$,

$$\xi_v(g) = \prod_{\alpha \in \mathcal{S}_v} \Xi_{\mathrm{PGL}_2(K_v)} \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}$$

where $\Xi_{\mathrm{PGL}_2(K_v)}$ is the Harish-Chandra function of $\mathrm{PGL}_2(K_v)$. If $K_v = \mathbb{C}$, set

$$\xi_v(g) = \prod_{\alpha \in \mathcal{S}_v} \Xi_{\mathrm{PGL}_2(\mathbb{C})} \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}^{n_\alpha}$$

where $n_\alpha = 1/2$ if α is a long root of $\mathbf{G}$, when $\mathbf{G}$ is locally isomorphic to $\mathrm{Sp}_{2n}(\mathbb{C})$, and $n_\alpha = 1$ for all other cases. We set $\xi_v = 1$ for $v \in \mathcal{T}$.

Since $0 < \xi_v(g_v) \leq 1$ for all $v \in R$ and $\xi_v(g_v) = 1$ for almost all v, the following function $\xi_{\mathbf{G}}$ is well defined:

Definition 3.1. *Define the function* $\xi_{\mathbf{G}} : \mathbf{G}(\mathbb{A}) \to (0,1]$ *by*

$$\xi_{\mathbf{G}}(g) = \prod_{v\in R} \xi_v(g_v) \quad \textit{for } g = (g_v)_v \in \mathbf{G}(\mathbb{A}).$$

Set

$$U_f = \prod_{v\in R_f} U_v, \quad U_\infty = \prod_{v\in R_\infty} U_v, \quad \text{and } U = U_f \times U_\infty. \tag{3.2}$$

Note also that $\xi_{\mathbf{G}}$ is bi-U-invariant.

For $v \in R - \mathcal{T}$, we set

$$\eta_v(kadk') := \prod_{\alpha\in\mathcal{S}_v} |\alpha(a)|_v$$

where $kadk' \in U_v A_v^+ \Omega_v U_v$ for all v with $K_v \neq \mathbb{C}$. As in the case of the definition of ξ_v, if $K_v = \mathbb{C}$ and for $kak' \in U_v A_v^+ U_v$, we set

$$\eta_v(kak') = \prod_{\alpha\in\mathcal{S}_v} |\alpha^{n_\alpha}(a)|_v$$

with the same n_α defined as before. If $v \in \mathcal{T}$, we set $\eta_v = 1$.

Lemma 3.3. *For any* $\epsilon > 0$*, there is a constant* $C_\epsilon > 0$ *such that for any* $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$*,*

$$\prod_{v\in R} \eta_v(g_v)^{-1/2} \leq \xi_{\mathbf{G}}(g) \leq C_\epsilon \cdot \prod_{v\in R} \eta_v(g_v)^{-1/2+\epsilon}. \tag{3.4}$$

In particular,

$$\xi_{\mathbf{G}}(g) \to 0 \quad \textit{as } g \to \infty \textit{ in } \mathbf{G}(\mathbb{A}).$$

Proof. For $v \in R - \mathcal{T}$, it follows from the explicit formula for Ξ_v (cf. [40, 3.8]) that for any $\epsilon > 0$, there is a constant $C_{v,\epsilon} > 0$ such that for any $g_v \in \mathbf{G}(K_v)$,

$$\eta_v(g_v)^{-1/2} \leq \xi_v(g_v) \leq C_{v,\epsilon} \cdot \eta_v(g_v)^{-1/2+\epsilon}.$$

Moreover one can take $C_{v,\epsilon} = 1$ for almost all v. This implies (3.4).

To see the second claim, first note that for any $g \in \mathbf{G}(\mathbb{A})$,

$$\xi_{\mathbf{G}}(g) \leq \xi_v(g_v) \leq C_{v,\epsilon} \cdot \eta_v(g_v)^{-1/2+\epsilon}. \tag{3.5}$$

Now suppose on the contrary that there exists a sequence $\{g_i \in \mathbf{G}(\mathbb{A})\}$ such that $g_i \to \infty$ and $\xi_{\mathbf{G}}(g_i) \not\to 0$. Then by passing to a subsequence we may assume either that there is a place $v \in R$ such that $g_{i,v} \to \infty$ in $\mathbf{G}(K_v)$ or that there exists a sequence $\{v_i \in R_f - S_0\}$ such that $g_{i,v_i} \notin U_{v_i}$ and $q_{v_i} \to \infty$. If $g_{i,v} \to \infty$ as $i \to \infty$, then $|\eta_v(g_{i,v})| \to \infty$ as $i \to \infty$ and hence $\xi_{\mathbf{G}}(g_i) \to 0$ by (3.4). Therefore the first case cannot happen.

In the second case, note that since $g_{i,v_i} \notin U_{v_i}$ and $\Omega_{v_i} = \{e\}$, we have $\eta_v(g_{i,v_i}) \geq q_{v_i}$ for each i. Hence by (3.5) for all i big enough,

$$\xi_{\mathbf{G}}(g_i) \leq C_{v_i,\epsilon} \cdot q_{v_i}^{-1/2+\epsilon} \leq q_{v_i}^{-1/2+\epsilon}.$$

This gives a contradiction since $q_{v_i} \to \infty$. □

Lemma 3.6. *Let $\iota : \mathbf{G} \to \mathrm{GL}_{\mathrm{N}}$ be a faithful representation defined over K with a unique maximal weight and H_ι be a height function on $\mathbf{G}(\mathbb{A})$ associated to ι. Then there exist $m \in \mathbb{N} - \{0\}$ and $C > 0$ such that*

$$\xi_{\mathbf{G}}(g) \leq C \cdot \mathrm{H}_\iota^{-1/m}(g) \quad \textit{for any } g \in \mathbf{G}(\mathbb{A}).$$

Proof. Let χ denote the highest weight of ι. Let $l \in \mathbb{N} - \{0\}$ be such that $\chi|_{A_v^+} \leq l \cdot \log_{q_v} \eta_v$ for each $v \in R$. Here $q_v = e$ if $v \in R_\infty$. Without loss of generality, we may assume

$$\mathrm{H}_v(\iota(a_v)) = q_v^{\chi(a_v)} \quad \text{for each } a_v \in A_v^+ \text{ and } v \in R.$$

Since $\eta_v(a_v) = q_v^{\log_{q_v} |\eta_v(a_v)|}$ for $a_v \in A_v^+$, we have for each $v \in R$,

$$\eta_v(a_v)^{-l} \leq \prod_v \mathrm{H}_v^{-1}(\iota(a_v)) \quad \text{for } a_v \in A_v^+.$$

By the continuity of H_v and the Cartan decomposition $\mathbf{G}(K_v) = U_v A_v^+ \Omega_v U_v$, there exists $r_v \geq 1$ such that

$$r_v^{-1}\, \mathrm{H}_v(\iota(a_v)) \leq \mathrm{H}_v(\iota(g_v)) \leq r_v\, \mathrm{H}_v(\iota(a_v))$$

for $g_v = k_v a_v d_v k_v' \in U_v A_v^+ \Omega_v U_v$. Since H_ι is invariant under a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and hence $\mathrm{H}_v \circ \iota$ is invariant under U_v for almost all $v \in R_f$, we can take $r_v = 1$ for almost all $v \in R_f$.

Therefore if $g = (g_v) \in \mathbf{G}(\mathbb{A})$ with $g_v = k_v a_v d_v k_v' \in U_v A_v^+ \Omega_v U_v$,

$$\begin{aligned}\prod_v \eta_v(g_v)^{-l} = \prod_v \eta_v(a_v)^{-l} &\leq \prod_v \mathrm{H}_v^{-1}(\iota(a_v)) \\ &\leq r_0 \prod_v \mathrm{H}_v^{-1}(\iota(g_v)) \leq r_0 \cdot \mathrm{H}_\iota^{-1}(g)\end{aligned}$$

where $r_0 = \prod_v r_v < \infty$.

Hence using Lemma 3.3, there exists $c_1 > 0$ such that for any $g = (g_v) \in \mathbf{G}(\mathbb{A})$,

$$\xi_{\mathbf{G}}^{4l}(g) \leq c_1 \cdot \prod_v \eta_v(g_v)^{-l} \leq c_1 \cdot r_0 \cdot \mathrm{H}_\iota^{-1}(g).$$

This proves the claim. □

Theorem 7.1 in [50] shows that for ay representation ι of $\mathbf{G}$ over K with a unique maximal weight, the height zeta function

$$\mathcal{Z}(s) := \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s} d\tau(g)$$

converges for $\Re(s) > a_\iota$ where a_ι is defined as in (1.3). In particular H_ι^{-1} belongs to $L^p(\mathbf{G}(\mathbb{A}))$ for any $p > a_\iota$. Hence as a corollary of [50, Theorem 7.1] using Lemma 3.6, we obtain the following:

Corollary 3.7. *There exists* $0 < p = p(\mathbf{G}) < \infty$ *such that* $\xi_{\mathbf{G}} \in L^p(\mathbf{G}(\mathbb{A}))$.

3.2. **Restriction of scalars functor.** We prove certain funtorial properties of $\xi_{\mathbf{G}}$ for the restriction of scalars functor introduced by Weil [59], which we will need later to reduce the dicussion on general almost K-simple groups to that on absolutely simple K-groups. We refer to [38, I. 3.1.4] and [8, Ch. 6] for the properties of the restrictions of scalar functor $R_{K/k}$ used in the following discussion. Suppose that k is a finite extension field of K and $\mathbf{G}'$ is a connected semisimple k-group. Then $\mathbf{G} = R_{k/K}\mathbf{G}'$ is a connected semisimple K-group.

Denote by $\{1 = \sigma_1, \cdots, \sigma_d\}$ the set of all distinct embeddings of k into the algebraic closure of K. Then there is a K-morphism $\mu : \mathbf{G} \to \mathbf{G}'$ such that the map

$$\mu^\circ = ({}^{\sigma_1}\mu, \cdots, {}^{\sigma_d}\mu) : \mathbf{G} \to {}^{\sigma_1}\mathbf{G}' \times \cdots \times {}^{\sigma_d}\mathbf{G}' \tag{3.8}$$

is a K-isomorphism. If $R^\circ_{k/K}$ denotes the inverse map to $\mu|_{\mathbf{G}(K)}$, then $R^\circ_{k/K} : \mathbf{G}'(k) \to \mathbf{G}(K)$ is a group isomorphism.

For each $v \in R_K$, denote by I_v the set of all valuations of k extending v. Then there is a natural K_v-isomorphism $f_v : \mathbf{G} \to \prod_{w\in I_v} R_{k_w/K_v}\mathbf{G}'$ and the isomorphisms $f_v^{-1} \circ R^\circ_{k_w/K_v} : \prod_{w\in I_v} \mathbf{G}'(k_w) \to \mathbf{G}(K_v)$, $v \in R_K$, $w \in I_v$, induce a topological group isomorphism, say, j, of the adele group $\mathbf{G}'(\mathbb{A}_k)$ to $\mathbf{G}(\mathbb{A}_K)$.

Lemma 3.9. *Let* $\epsilon > 0$. *Then there are constants* $C_\epsilon \geq 1$ *such that for any* $g \in \mathbf{G}'(\mathbb{A}_k)$,

$$C_\epsilon^{-1} \cdot \xi_{\mathbf{G}}(j(g))^{1+\epsilon} \leq \xi_{\mathbf{G}'}(g) \leq C_\epsilon \cdot \xi_{\mathbf{G}}(j(g))^{1-\epsilon}.$$

Proof. Fix $v \in R_K$. The set I_v parametrizes the set, say, of all distinct embeddings σ of k into $\bar{K}_v$ which are non-conjugate over K_v, in the way that $w \in I_v$ corresponds to σ with $k_w = \sigma(k)K_v$. For each embedding $w \in I_v$, we denote by J_w the set of all embeddings τ of k into $\bar{K}_v$ such that $k_w = \tau(k)K_v$. Fix $w \in I_v$. Let A'_w be the group of k_w-points of a maximal k_w-split A' torus of $\mathbf{G}'$, $\Phi(\mathbf{G}'(k_w), A'_w)$ the set of non-multipliable roots, and $\mathcal{S}_w \subset \Phi(\mathbf{G}'(k_w), A'_w)$ be a maximal strongly orthogonal system used in the definition of $\xi_{\mathbf{G}'}$. Then $\prod_{w\in I_v}(R_{k_w/K_v}A')$ is a maximal K_v-torus of $\mathbf{G}$ and its maximal K_v-split subtorus is a maximal K_v-split torus of $\mathbf{G}$ [8, Ch. 6]. For each $w \in I_v$, let $B(w)$ denote the group of K_v-points of the maximal K_v-split torus of $R_{k_w/K_v}A'$ and set $A_v = \prod_{w\in I_v} B(w)$. We can identify each $B(w)$ with $\{({}^\tau a_w)_{\tau\in J_w} : a_w \in A'(K_v)\}$, and $\Psi_w := \Phi((R_{k_w/K_v}\mathbf{G}')(K_v), B(w))$ with $\{\alpha_w : \alpha \in \Phi(\mathbf{G}'(k_w), A'_w)\}$ where $\alpha_w(({}^\tau a_w)_{\tau\in J_w}) = \prod_{\tau\in J_w} {}^\tau\alpha(a_w)$. Hence $\{\alpha_w : \alpha \in \mathcal{S}_w\}$ is a maximal strongly orthogonal system of Ψ_w.

On the other hand, for any $\alpha \in \mathcal{S}_w$,

$$|\alpha(a_w)|_w = |\prod_{\tau\in J_w} {}^\tau\alpha(a_w)|_v.$$

Since $\prod_{w\in I_v}\mathcal{S}_w$ is a maximal strongly orthogonal system for $\Phi(\mathbf{G}(K_v),A_v)$, if $a=((^\tau a_w)_{\tau\in J_w})_{w\in I_v}\in A_v$,

$$\eta_{\mathbf{G}(K_v)}(a)=\prod_{(\alpha_w)_{w\in I_v}\in\prod_{w\in I_v}\mathcal{S}_w}|\alpha_w((^\tau a_w)_{\tau\in J_w})|_v=\prod_{w\in I_v}\prod_{\alpha_w\in\mathcal{S}_w}|\alpha(a_w)|_w=\prod_{w\in I_v}\eta_{\mathbf{G}'(k_w)}(a_w).$$

Hence for all $g\in\mathbf{G}'(\mathbb{A}_k)$,

$$\prod_{v\in R_K}\eta_{\mathbf{G}(K_v)}(j(g))=\prod_{w\in R_k}\eta_{\mathbf{G}'(k_w)}(g).$$

By Lemma 3.3, this proves the claim. □

3.3. **Uniform bound for matrix coefficients of $\mathbf{G}(\mathbb{A})$.** Let $W_f\subset\mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. Write $W_v=W_f\cap\mathbf{G}(K_v)$ for each $v\in R$. Then $W_v=U_v$ for almost all $v\in R_f$. For each $v\in R_f$, by [5], there exists $d_{W_v}<\infty$ such that for any irreducible unitary representation ρ of $\mathbf{G}(K_v)$, the dimension of W_v-invariant vectors of ρ is at most d_{W_v}. Moreover $d_{W_v}=1$ for almost all $v\in R$ by [57, 3.3.3] and [26, Corollary 1]. Hence the following number is well-defined:

$$d_{W_f}:=\prod_{v\in R_f}d_{W_v}<\infty.$$

The notation $\mathbf{G}(K_v)^+$ denotes the normal subgroup of $\mathbf{G}(K_v)$ generated by all unipotent subgroups of $\mathbf{G}(K_v)$.

Theorem 3.10. *Let $\mathbf{G}$ be a connected absolutely almost simple K-group with K-rank at least 2. Let W_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Let π be any unitary representation of $\mathbf{G}(\mathbb{A})$ without no non-trivial $\mathbf{G}(K_v)^+$-invariant vector for every $v\in R$. Then for any U_∞-finite and W_f-invariant unit vectors x and y,*

$$|\langle\pi(g)x,y\rangle|\le d_0\cdot c_{W_f}\cdot(\dim\langle U_\infty x\rangle\cdot\dim\langle U_\infty y\rangle)^{(r+1)/2}\cdot\xi_{\mathbf{G}}(g)\quad\textit{for all } g\in\mathbf{G}(\mathbb{A})\tag{3.11}$$

where $c_{W_f}:=d_{W_f}\cdot\prod_v[U_v:U_v\cap W_v]\cdot(\max_{d\in\Omega_v}[U_v:dU_vd^{-1}])$ and $d_0,r\ge 1$ depend only on $\mathbf{G}$. Moreover if $\mathbf{G}(K_v)\not\cong\mathrm{Sp}_{2n}(\mathbb{C})$ locally for any $v\in R_\infty$, $d_0=1$ and $r=1$.

If $\mathbf{G}$ is a connected almost K-simple adjoint (simply connected) K-group with K-rank at least 2, and π has no non-trivial $\mathbf{L}(K_v)^+$-invariant vectors for any connected K_v-normal subgroup $\mathbf{L}$ of $\mathbf{G}$, then (3.11) holds with $\xi_{\mathbf{G}}$ replaced by $\xi_{\mathbf{G}}^{1-\epsilon}$ for any $\epsilon>0$.

As a corollary, we obtain the adelic version of Howe-Moore theorem [33] on the vanishing of matrix coefficients:

Corollary 3.12. *Let $\mathbf{G}$ and π be as in Theorem 3.10. Then for any vectors x and y,*

$$\langle\pi(g)x,y\rangle\to 0\quad\textit{as } g\to\infty\textit{ in }\mathbf{G}(\mathbb{A}).$$

Proof. It is easy to see that it suffices to prove the claim for a dense subset of vectors in the Hilbert space V associated to π. Considering the restriction $\tilde\pi:=\pi|_U$ to U, V decomposes into a direct sum of irreducible unitary representations of the compact

group U each of which is finite dimensional by Peter-Weyl theorem. Hence the set V_0 of U-finite vectors is dense in V. Now if $x, y \in V_0$, then x and y are invariant under a finite index subgroup W_f of U_f. Hence by applying the above theorem, we obtain that for some constant $c_0 > 0$,

$$|\langle \pi(g)x, y\rangle| \le c_0 \cdot \xi_{\mathbf{G}}(g) \quad \text{for all } g \in \mathbf{G}(\mathbb{A}).$$

Since $\xi_{\mathbf{G}}(g) \to 0$ as $g \to \infty$, this implies the claim. $\square$

The proof of Theorem 3.10 is based on theorems in [40]. More precisely, recall:

Theorem 3.13. [40, Theorem 1.1-2] *Suppose that the K_v-rank of $\mathbf{G}$ is at least 2. Let π_v be a unitary representation of $\mathbf{G}(K_v)$ without any non-trivial $\mathbf{G}(K_v)^+$-invariant vectors. Then for any U_v-finite unit vectors x and y,*

$$|\langle \pi_v(g)x, y\rangle| \le d_v \cdot c_v \cdot (\dim\langle U_v x\rangle \cdot \dim\langle U_v y\rangle)^{r_v/2} \cdot \xi_v(g) \quad \textit{for any } g \in \mathbf{G}(K_v)$$

where $c_v = \max_{d\in\Omega_v}[U_v : dU_v d^{-1}]$ and $d_v, r_v \ge 1$ depend only on $\mathbf{G}(K_v)$. Moreover whenever $\mathbf{G}(K_v) \ncong \mathrm{Sp}_{2n}(\mathbb{C})$ locally, $d_v = 1$ and $r_v = 1$.

In the case when $\mathbf{G}(K_v) \cong \mathrm{Sp}_{2n}(\mathbb{C})$ locally, the above theorem was stated only for U_v-invariant vectors in [40]. However if we replace Proposition 2.7 in [40] by the remark following it, the same proof works for the above claim.

Proof of Theorem 3.10. We first assume that $\mathbf{G}$ is absolutely almost simple. For $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$, choose a finite subset S_g of places containing

$$\{v \in R_f : g_v \notin U_v\} \cup R_\infty.$$

Note that for $v \in R - S_g$, we have $g_v \in U_v$ and hence $\xi_v(g_v) = 1$. Therefore for $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$,

$$\xi(g) = \prod_{v\in S_g} \xi_v(g_v).$$

Let $G_g = \prod_{v\in S_g} \mathbf{G}(K_v)$ and $W_g = \prod_{v\in S_g\cap R_f} W_v$. As a G_g representation, π has a Hilbert integral decomposition:

$$\pi = \int_{z\in Z_g} \oplus^{m_z} \rho_z \, d\nu(z)$$

where Z_g is the unitary dual of G_g and ρ_z is irreducible, m_z is a multiplicity for each $z \in Z_g$ and ν is a measure on Z_g (see [22] or [61, Section 2.3]). We may assume that for all z, ρ_z has no $\mathbf{G}(K_v)^+$-invariant vector (see [61, Prop. 2.3.2]).

If we write $\mathcal{L}_z = \oplus^{m_z}\rho_z$, $x = \int x_z d\nu(z)$ and $y = \int y_z d\nu(z)$ with

$$x_z = \sum_{i=1}^{m_z} x_{zi} \quad \text{and} \quad y_z = \sum_{i=1}^{m_z} y_{zi} \in \mathcal{L}_z,$$

we have

$$\langle x, y\rangle = \int_{Z_g} \sum_{i=1}^{m_z} \langle x_{zi}, y_{zi}\rangle \, d\nu(z).$$

It follows from the definition of a Hilbert direct integral that

$$\dim\langle U_\infty x_{zi}\rangle \le \dim\langle U_\infty x_z\rangle \le \dim\langle U_\infty x\rangle,$$

x_{zi} is W_g-invariant for almost all z and all i, and similarly for y. Without loss of generality, we assume the above holds for all z. We claim that

(3.14)

$$|\langle \rho_z(g) x_{zi}, y_{zi}\rangle| \le c_{W_f} \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \|x_{zi}\| \cdot \|y_{zi}\|$$

where $r = \max_v r_v$ and $c_0 = d_{W_f} \prod_v (c_v \cdot [U_v : U_v \cap W_v]) < \infty, d_0 = \prod_v d_v < \infty$ with c_v, d_v, r_v as in Theorem 3.13. By [5], we may write $\rho_z = \otimes_{v\in S_g} \rho_{z(v)}$ where $\rho_{z(v)}$ is an irreducible representation of $\mathbf{G}(K_v)$ without no non-trivial $\mathbf{G}(K_v)^+$-invariant vectors. Since the finite linear combinations of pure tensor vectors are dense, it suffices to prove (3.14) assuming x_{zi} and y_{zi} are finite sums of pure tensors. Hence we can write

$$x_{zi} = \sum_j \bigotimes_{v\in S_g} x_{zij(v)} \ ; \ y_{zi} = \sum_k \bigotimes_{v\in S_g} y_{zik(v)}$$

where for each $v \in S_g$, $x_{zij(v)}$ (resp. $y_{zik(v)}$) are mutually orthogonal and the number of summands for x_{zi} (resp. y_{zi}) is at most $\dim\langle U_\infty x\rangle \cdot d_{W_f}$ (resp. $\dim\langle U_\infty y\rangle \cdot d_{W_f}$). Hence by Cauchy-Schwarz inequality, for $x_{zij} = \prod_{v\in S_g} x_{zij(v)}$ and $y_{zij} = \prod_{v\in S_g} y_{zij(v)}$

$$\sum_j \|x_{zij}\| \le (\dim\langle U_\infty x\rangle \cdot d_{W_f})^{1/2} \|x_{zi}\|; \text{ and } \sum_k \|y_{zik}\| \le (\dim\langle U_\infty y\rangle \cdot d_{W_f})^{1/2} \|y_{zi}\|.$$

Since for $v \in R_f$

$$\dim\langle U_v x\rangle \le [U_v : W_v \cap U_v] \quad \text{and} \quad \dim\langle U_v y\rangle \le [U_v : W_v \cap U_v],$$

by Theorem 3.13, we have for $c_0 = \prod_v c_v$,

(3.15)

$$\begin{aligned}
&|\langle \rho_z(g) x_{zi}, y_{zi}\rangle| \le \sum_{j,k} \prod_{v\in S_g} |\langle \rho_{z(v)}(g_v) x_{zij(v)}, y_{zik(v)}\rangle| \\
&\le c_0 \cdot d_0 \cdot \prod_{v\in S_g} \xi_v(g_v) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{r/2} (\prod_{v\in R_f} [U_v : W_v \cap U_v]) \cdot \left(\sum_{j,k} \|x_{zij}\| \cdot \|y_{zik}\|\right) \\
&\le c_0 \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \prod_{v\in R_f} [U_v : W_v \cap U_v]) \cdot d_{W_f} (\|x_{zi}\| \cdot \|y_{zi}\|) \\
&= c_{W_f} \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot (\|x_{zi}\| \cdot \|y_{zi}\|)
\end{aligned}$$

proving (3.14). Therefore again by Cauchy-Schwarz inequality,

(3.16)

$$\begin{aligned}|\langle(\oplus^{m_z}\rho_z)(g)(x_z), y_z\rangle| &\le \sum_i |\langle\rho_z(g)x_{zi}, y_{zi}\rangle| \\ &\le c_{W_f}\cdot d_0\cdot\xi_{\mathbf{G}}(g)\cdot(\dim\langle U_\infty x\rangle\cdot\dim\langle U_\infty y\rangle)^{(r+1)/2}\cdot\|x_z\|\cdot\|y_z\|.\end{aligned}$$

By integrating over Z_g, we obtain (3.11).

Since $\mathbf{G}$ is adjoint (resp. simply connected), there exists a finite separable extension k of K and a connected adjoint (resp. simply connected) absolutely almost simple k-group $\mathbf{G}'$ such that $\mathbf{G} = R_{k/K}\mathbf{G}'$ by [56, 3.1.2]. Then the topological isomorphism $j^{-1} : \mathbf{G}(\mathbb{A}_K) \to \mathbf{G}'(\mathbb{A}_k)$ described prior to Lemma 3.9 maps $\mathbf{G}(K_v)$ and $\mathbf{G}(K_v)^+$ to $\prod_{w\in I_v}\mathbf{G}'(k_w)$ and $\prod_{w\in I_v}\mathbf{G}'(k_w)^+$ respectively. Let π be a representation on $\mathbf{G}(\mathbb{A})$ satisfying the hypothesis. Then the representation, say, π', on $\mathbf{G}'(\mathbb{A}_k)$ induced by π has no $\mathbf{G}'(k_w)^+$-invariant vectors for each $w \in R_k$. Since the k-rank of $\mathbf{G}'$ is equal to the K-rank of $\mathbf{G}$ and hence is at least 2, by the assertion already proved for $\mathbf{G}'(\mathbb{A}_k)$, we deduce that for any U_∞-finite and W_f-invariant x, y,

$$\begin{aligned}|\langle\pi(g)x, y\rangle| &= |\langle\pi'(j^{-1}(g))x, y\rangle| \\ &\le d_0\cdot c_{W_f}\cdot(\dim\langle U_\infty x\rangle\cdot\dim\langle U_\infty y\rangle)^{(r+1)/2}\cdot\xi_{\mathbf{G}'}(j^{-1}(g)) \quad \text{for all } g\in\mathbf{G}(\mathbb{A}).\end{aligned}$$

By (3.19), we may replace $\xi_{\mathbf{G}'}(j^{-1}(g))$ by $\xi_{\mathbf{G}}(g)^{1-\epsilon}$, finishing the proof. □

3.4. **Automorphic bound for $\mathbf{G}(\mathbb{A})$.** If $\mathbf{G}$ has K-rank at most one, the analogue of Theorem 3.10 does not hold in general. However if we look at those infinite dimensional representations occurring in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, we still obtain a similar upper bound.

We first state the following conjecture:

Conjecture 3.17. *Let $\mathbf{G}$ be a connected absolutely almost simple K-group. Let W_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then for any $U_\infty\times W_f$-invariant unit vectors $f, h\in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$|\langle f, g.h\rangle| \le c_{W_f}\cdot\xi_{\mathbf{G}}(g) \quad \textit{for all } g\in\mathbf{G}(\mathbb{A})$$

where $c_{W_f} > 0$ is a constant depending only on $\mathbf{G}$ and W_f.

The above holds for groups of K-rank at least 2 by Theorem 3.10. For $\mathbf{G} = \mathrm{PGL}_2$, Conjecture 3.17 is essentially equivalent to the Ramanujan conjecture. We will prove a weaker statement of Conjecture 3.17 where the function $\xi_{\mathbf{G}}$ is replaced by a function $\tilde{\xi}_{\mathbf{G}}$ with slower decay such that $\xi_{\mathbf{G}} \le \tilde{\xi}_{\mathbf{G}} \le \xi_{\mathbf{G}}^{1/2}$.

Definition 3.18. *Let $\mathbf{G}$ be a connected almost K-simple group. For each $v\in R$, write $\mathbf{G}$ as an almost direct product $\mathbf{G}_v^1\mathbf{G}_v^2$ where $\mathbf{G}_v^1$ is the maximal semisimple normal K_v-subgroup of $\mathbf{G}$ such that every simple normal K_v-subgroup of $\mathbf{G}_v^1$ has K_v-rank one.*

Note that $\mathbf{G}_v^2$ is then the maximal semisimple normal K_v-subgroup of $\mathbf{G}$ without any K_v-normal subgroup of rank zero or one. We define a function $\tilde{\xi}_{\mathbf{G}} : \mathbf{G}(\mathbb{A}) \to (0, 1]$ by

$$\tilde{\xi}_{\mathbf{G}} := \prod_{v \in R} \left(\xi^{1/2}_{\mathbf{G}_v^1(K_v)} \cdot \xi_{\mathbf{G}_v^2(K_v)} \right).$$

If $\mathbf{G}$ is absolutely almost simple and $R_1 := \{v \in R : \mathrm{rank}_{K_v}(\mathbf{G}) = 1\}$. then

$$\tilde{\xi}_{\mathbf{G}} = \prod_{v \in R_1} \xi^{1/2}_{\mathbf{G}(K_v)} \cdot \prod_{v \in R - R_1} \xi_{\mathbf{G}(K_v)}.$$

If $\mathbf{G} = R_{K/k}\mathbf{G}'$ for some finite extension field k and for a connected absolutely almost simple k-group $\mathbf{G}'$, then, for any $w \in R_k$ extending $v \in R_K$, the k_w-rank of a connected simple k_w-subgroup $\mathbf{H}'$ of $\mathbf{G}'$ is equal to the K_v-rank of the K_v-subgroup $R_{k_w/K_v}\mathbf{H}'$ of $\mathbf{G}$ (cf. [38, Ch I. 3.1]). Using this, the proof of Lemma 3.9 also shows that there is $C_\epsilon > 1$ such that for any $g \in \mathbf{G}'(\mathbb{A}_k)$

$$C_\epsilon^{-1} \cdot \tilde{\xi}_{\mathbf{G}}(j(g))^{1+\epsilon} \le \tilde{\xi}_{\mathbf{G}'}(g) \le C_\epsilon \cdot \tilde{\xi}_{\mathbf{G}}(j(g))^{1-\epsilon}. \tag{3.19}$$

where j denotes the topological group isomorphism from $\mathbf{G}'(\mathbb{A}_k)$ to $\mathbf{G}(\mathbb{A})$ described prior to Lemma 3.9.

Theorem 3.20 (Automorphic bounds)**.** *Let $\mathbf{G}$ be a connected absolutely almost simple K-group, For a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$, there exist $r = r(\mathbf{G}) \ge 1$ and $c_{W_f} > 0$ such that for any U_∞-finite and W_f-invariant unit vectors $x, y \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$|\langle x, g.y\rangle| \le c_{W_f} \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \tilde{\xi}_{\mathbf{G}}(g) \quad \text{for all } g \in \mathbf{G}(\mathbb{A}).$$

One can take $r = 1$ provided for any $v \in R$, $\mathbf{G}(K_v)$ has no subgroup locally isomorphic to $\mathrm{Sp}_{2n}(\mathbb{C})$ $(n \ge 2)$ locally.

If $\mathbf{G}$ is a connected almost K-simple adjoint (simply connected) K-group, then the above inequality holds with $\tilde{\xi}_{\mathbf{G}}$ replaced by $\tilde{\xi}_{\mathbf{G}}^{1-\epsilon}$ for any $\epsilon > 0$.

Recall that for unitary representations ρ_1 and ρ_2 of $\mathbf{G}(K_v)$, ρ_1 is said to be weakly contained in ρ_2 if every diagonal matrix coefficients of ρ_1 can be approximated uniformly on compact subsets by convex combinations of diagonal matrix coefficients of ρ_2. For each $v \in R$, denote by $\hat{\mathbf{G}}_v$ the unitary dual of $\mathbf{G}(K_v)$ and by $\hat{\mathbf{G}}_v^{\mathrm{Aut}} \subset \hat{\mathbf{G}}_v$ the automorphic dual of $\mathbf{G}(K_v)$ as defined in the introduction. The following theorem was first obtained by Burger and Sarnak for v archimedean [12, Theorem 1.1] and generalized by Clozel and Ullmo to all v [18, Theorem 1.4].

Theorem 3.21. *Let $\mathbf{G}$ be a connected absolutely almost simple K-group. Let $\mathbf{H} \subset \mathbf{G}$ be a connected semisimple K-subgroup. Then for any $v \in R$ and for any $\rho_v \in \hat{\mathbf{G}}_v^{\mathrm{Aut}}$, any irreducible representation of $\mathbf{H}(K_v)$ weakly contained in $\rho_v|_{\mathbf{H}(K_v)}$is contained in $\hat{\mathbf{H}}_v^{\mathrm{Aut}}$.*

Lemma 3.22. *For any $v \in R$ such that $\mathbf{G}(K_v)$ is non-compact, $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ has no non-zero $\mathbf{G}(K_v)^+$-invariant function.*

Proof. Let $\mathcal{L}_v$ denote the set of $f \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ fixed by $\mathbf{G}(K_v)^+$. We need to show that $\mathcal{L}_v = \{0\}$. Let $\mathbf{G}^{\{v\}}$ denote the subgroup of $\mathbf{G}(\mathbb{A})$ consisting of elements whose v-component is trivial. Consider the family of continuous functions $f \in C_c(\mathbf{G}^{\{v\}})$ of the form $f = \prod_{w\in R-\{v\}} f_w$ where each f_w is a continuous function of $\mathbf{G}(K_w)$ such that $f_w|_{U_w} = 1$ for almost all w. By considering the convolutions with these functions, we obtain a dense family of the continuous functions belonging to $\mathcal{L}_v$. Hence it suffices to show that any continuous function $f \in \mathcal{L}_v$ is trivial. Let $f \in \mathcal{L}_v$ be continuous. Let $\tilde{\mathbf{G}}$ be the simply connected cover of $\mathbf{G}$ and denote by $\mathrm{pr} : \tilde{\mathbf{G}} \to \mathbf{G}$ the covering map. Consider the projection map

$$\tilde{\mathbf{G}}(K)\backslash\tilde{\mathbf{G}}(\mathbb{A}) \to \mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}).$$

Let $\tilde{f}$ be the pull back of f. Since the image of $\tilde{\mathbf{G}}(K_v)$ under the map pr is $\mathbf{G}(K_v)^+$, the function $\tilde{f}$ is left $\tilde{\mathbf{G}}(K)$-invariant and right $\tilde{\mathbf{G}}(K_v)$-invariant. On the other hand, the strong approximation property implies that $\tilde{\mathbf{G}}(K)\tilde{\mathbf{G}}(K_v)$ is dense in $\tilde{\mathbf{G}}(\mathbb{A})$ (cf. [44, Theorem 7.12]). Therefore $\tilde{f}$ is constant, and hence f factors through the image of $\tilde{\mathbf{G}}(\mathbb{A})$ in $\mathbf{G}(\mathbb{A})$. Since $\mathbf{G}(\mathbb{A})/\tilde{\mathbf{G}}(\mathbb{A})$ is abelian, $L^2(\mathbf{G}(K)\tilde{\mathbf{G}}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A}))$ is a sum of automorphic characters of $\mathbf{G}(\mathbb{A})$. Since f is orthogonal to Λ, $f = 0$. □

Proof of Theorem 3.20. We first treat the case when $\mathbf{G}$ is absolutely almost simple. The case when K-rank is at least 2 follows from Theorem 3.10 and Lemma 3.22. Suppose first that $\mathbf{G}$ has K-rank one. By [17, Theorem 3.4], for $v \in R_1$, any infinite dimensional $\rho_v \in \hat{\mathbf{G}}_v^{\mathrm{Aut}}$, and U_v-finite vectors x_v, y_v,

$$|\langle\rho_v(g)(x_v), y_v\rangle| \le c_v \cdot \xi_v(g)^{1/2} \cdot (\dim\langle U_v x_v\rangle \cdot \dim\langle U_v y_v\rangle)^{1/2} \tag{3.23}$$

for any $g \in \mathbf{G}(K_v)$. Combining this with Theorem 3.13, we can derive the desired bound by the same argument as in the proof of Theorem 3.10.

Now suppose $\mathbf{G}$ is K-anisotropic. For $v \in R_1$, we claim (3.23) holds. In [15], it is analyzed what kind of ρ_v occurs in this situation, and this is the main case which was not known before Clozel's work. We give a brief summary. If $R_1 \neq \emptyset$, it follows from the classification theorem by Tits [56] that $\mathbf{G}$ is of Dynkin type $\mathcal{A}$. [15, Theorem 1.1] says that there exists a K-embedding of K-subgroup $\mathbf{H}$ of type $\mathcal{A}$ such that $\mathbf{H}$ has K_v-rank one whenever $v \in R_1$. Let $v \in R_1$. Then up to isogeny, one has either that $\mathbf{H} = \mathrm{PGL}_1(D)$ for a quaternion algebra D over K and $\mathbf{H} = \mathrm{PGL}_2$ over K_v, or $\mathbf{H} = \mathrm{PGU}(D, *)$ for a division algebra D of prime degree d over a quadratic extension k of K with a second kind involution $*$, and $\mathbf{H} = \mathrm{PGU}(n-1, 1)$ over K_v (with $n \ge 3$). In the former case one uses the Jacquet-Langlands correspondence [34] to transfer the Gelbart-Jacquet automorphic bound of PGL_2 to $\mathbf{H}(K_v)$ via Theorem 3.21. In the second case which is hardest, by the base changes obtained by Rogawski [45] and Clozel [16], we can use the bound of $\mathrm{PGL}_n(F_w)$ given by Theorem 3.13 to get a bound for $\mathbf{H}(K_v)$ where w is a place of k lying above v and k_w is a quadratic

extension of K_v. This proves the claim. Combining with Theorem 3.13 for those places $v \in R - (R_1 \cup \mathcal{T})$ as in the proof of Theorem 3.10, we obtain the desired bound.

Now for the case when $\mathbf{G}$ is adjoint (resp. simply connected) almost K-simple, the same argument used in the proof of Theorem 3.10 applies, since (in the notation therein) the topological isomorphism $j : \mathbf{G}(\mathbb{A}_K) \to \mathbf{G}'(\mathbb{A}_k)$ induces an equivariant isometry between $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ and $L^2_{00}(\mathbf{G}'(k)\backslash\mathbf{G}'(\mathbb{A}_k))$. □

3.5. **From U_∞-finite vectors to smooth vectors.** In Theorem 3.20, we can relax U_∞-finite conditions to smooth conditions provided we replace the L^2-norms by L^2-Sobolev norms. For a precise formulation, let $X_1, \cdots, X_m$ be an orthonormal basis of the Lie algebra $\mathrm{Lie}(U_\infty)$ with respect to an Ad-invariant scalar product. Then the elliptic operator

$$\mathcal{D} := 1 - \sum_{i=1}^{m} X_i^2 \tag{3.24}$$

lies in the center of the universal enveloping algebra of $\mathrm{Lie}(U_\infty)$. We say a function f on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ is smooth if f is invariant under some compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and smooth for the action of $\mathbf{G}_\infty$.

Theorem 3.25. *Let $\mathbf{G}$ be a connected almost absolutely simple K-group. and W_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then there exist an explicit $l \in \mathbb{N}$ and $c_{W_f} > 0$ such that for any W_f-invariant smooth functions $\varphi, \psi \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ with $\|\mathcal{D}^l(\varphi)\| < \infty$ and $\|\mathcal{D}^l(\psi)\| < \infty$,*

$$|\langle \varphi, g.\psi \rangle| \le c_{W_f} \cdot \tilde{\xi}_{\mathbf{G}}(g) \cdot \|\mathcal{D}^l(\varphi)\| \cdot \|\mathcal{D}^l(\psi)\| \quad \text{for all } g \in \mathbf{G}(\mathbb{A}).$$

If $\mathbf{G}$ is a connected almost K-simple adjoint (simply connected) K-group, then the above inequality holds with $\tilde{\xi}_{\mathbf{G}}$ replaced by $\tilde{\xi}_{\mathbf{G}}^{1-\epsilon}$ for any $\epsilon > 0$.

Proof. Deducing this from Theorem 3.20 is quite standard in view of the results of Harish-Chandra explained in [58, Ch 4]. We give a sketch of the proof. Denote by π the representation $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Then $\pi = \oplus_{\nu \in \hat{U}_\infty} \pi_\nu$ where π_ν is the ν-isotypic component of π and $\mathcal{D}$ acts as a scalar, say, c_ν on each π_ν. We write $\varphi = \sum_{\nu \in \hat{U}_\infty} \varphi_\nu$ and $\psi = \sum_{\nu \in \hat{U}_\infty} \psi_\nu$. One has $\|\varphi_\nu\| = c_\nu^{-l}\|\mathcal{D}^l\varphi_\nu\|$ and similarly for ψ. Then

$$|\langle \varphi, g.\psi \rangle| \le \sum_{(\nu_1,\nu_2) \in \hat{U}_\infty \times \hat{U}_\infty} |\langle \varphi_{\nu_1}, g.\psi_{\nu_2} \rangle|$$

Using Theorem 3.20, we then obtain

$$\begin{aligned}
&|\langle \varphi, g.\psi\rangle| \\
&\le c_{W_f}\cdot \tilde{\xi}_{\mathbf{G}}(g)\left(\sum_{\nu\in\hat{U}_\infty}\|\varphi_\nu\|\dim\langle U_\infty\varphi_\nu\rangle^{(r+1)/2}\right)\left(\sum_{\nu\in\hat{U}_\infty}\|\psi_\nu\|\dim\langle U_\infty\psi_\nu\rangle^{(r+1)/2}\right)\\
&\le c_{W_f}\cdot \tilde{\xi}_{\mathbf{G}}(g)\cdot\|\mathcal{D}^l(\varphi)\|\cdot\|\mathcal{D}^l(\psi)\|\cdot\sum_{\nu\in\hat{U}_\infty} {c_\nu}^{-2l}\dim(\nu)^{r+1}
\end{aligned}$$

Now if $l\in\mathbb{N}$ is sufficiently large, then $\sum_\nu c_\nu^{-2l}\dim(\nu)^{r+1}<\infty$ [58, Lemma 4.4.2.3]. This proves the claim. $\square$

3.6. **From K-simple groups to semisimple groups.** If $\mathbf{G}$ is a connected semisimple K-group, we say that a sequence $\{g_i\in\mathbf{G}(\mathbb{A})\}$ tends to infinity strongly if for any non-trivial connected simple normal K-subgroup $\mathbf{H}$ of $\mathbf{G}$, $p_{\mathbf{H}}(g_i)$ tends to ∞ as $i\to\infty$, where $p_{\mathbf{H}}:\mathbf{G}(\mathbb{A})\to\mathbf{G}(\mathbb{A})/\mathbf{H}(\mathbb{A})$ denotes the canonical projection.

Theorem 3.26 (Mixing for $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$)**.** *Let $\mathbf{G}$ be a product of connected almost K-simple K-groups. Then for any $\varphi,\psi\in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$\langle\varphi, g.\psi\rangle\to 0$$

as $g\in\mathbf{G}(\mathbb{A})$ tends to infinity strongly.

Proof. Write $\mathbf{G}=\mathbf{G}_1\times\cdots\times\mathbf{G}_m$ where each $\mathbf{G}_i$ is a connected absolutely almost simple K-group. By Theorem 3.20 and Peter-Weyl theorem (cf. Corollary 3.12), for each $1\le i\le m$, and for any $\varphi_i,\psi_i\in L^2_{00}(\mathbf{G}_i(K)\backslash\mathbf{G}_i(\mathbb{A}))$,

$$\langle\varphi_i, g_i.\psi_i\rangle\to 0 \tag{3.27}$$

as $g_i\to\infty$ in $\mathbf{G}_i(\mathbb{A})$.

Consider $\otimes_{i=1}^m L^2(\mathbf{G}_i(K)\backslash\mathbf{G}_i(\mathbb{A}))$ as a subset of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. The finite sums of the functions of the form $\psi=\otimes_{i=1}^m\psi_i\in L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, $\psi_j\in L^2(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$, such that for at least one j, $\psi_j\in L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$ form a dense subset of the space $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Hence it suffices to prove the claim for $\varphi=\otimes_{i=1}^m\varphi_i$ and $\psi=\otimes_{i=1}^m\psi_i$ of such type. Suppose $\psi_j\in L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$ for some $1\le j\le m$. If $g=(g_1,\cdots,g_m)$ with $g_i\in\mathbf{G}_i(\mathbb{A})$, then

$$|\langle\varphi, g.\psi\rangle|=\prod_{i=1}^m|\langle\varphi_i,g_i.\psi_i\rangle|\le|\langle\varphi_j,g_j.\psi_j\rangle|\cdot(\prod_{i\ne j}\|\varphi_i\|\cdot\|\psi_i\|).$$

If φ_j' denotes the projection of φ_j to $L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$, then

$$\langle\varphi_j, g_j.\psi_j\rangle=\langle\varphi_j', g_j.\psi_j\rangle.$$

Since $g\to\infty$ strongly and hence $g_j\to\infty$, we obtain $\langle\varphi_j', g_j.\psi_j\rangle\to 0$ by (3.27). This proves the claim. $\square$

By the following proposition, the above theorem applies to connected semisimple adjoint (simply connected) K-groups.

Proposition 3.28. [56, 3.1.2] *Any connected semisimple adjoint (resp. simply connected) K-group decomposes into a direct product of adjoint (resp. simply connected) almost K-simple K-groups.*

3.7. **Equidistribution of Hecke points.** In this subsection which is not needed in the rest of the paper, we explain applications of the adelic mixing in the equidistribution problems of Hecke points considered in [17]. Let $K = \mathbb{Q}$. Let S be a finite set of primes including the archimedean prime ∞. If Γ is an S-arithmetic subgroup of $\mathbf{G}_S$ (here $\mathbb{Q}_\infty = \mathbb{R}$) and $a \in \mathbf{G}(\mathbb{Q})$, then the Hecke operator T_a on $L^2(\Gamma\backslash\mathbf{G}_S)$ is defined by

$$\mathrm{T}_a(\psi)(g) = \frac{1}{\deg(a)} \sum_{x\in\Gamma\backslash\Gamma a\Gamma} \psi(xg)$$

where $\deg(a) = \#\Gamma\backslash\Gamma a\Gamma$. Theorem 1.11 extends the main result in [17] where some cases of $\mathbb{Q}$-anisotropic groups were excluded (see [25]). In fact, the following corollary immediately follows from Theorem 3.20 and Proposition 2.6 in [17]:

Corollary 3.29. *Let $\mathbf{G}$ be a connected simply connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group and S a finite set of primes including ∞. Suppose that $\mathbf{G}_S$ is non-compact. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an S-congruence subgroup of $\mathbf{G}_S$. For any $\epsilon > 0$, there exists a constant $c = c(\Gamma, \epsilon) > 0$ such that*

$$\| \mathrm{T}_a \| \le c \cdot \tilde{\xi}_{\mathbf{G}}^{1-\epsilon}(a) \quad \text{for any } a \in \mathbf{G}(\mathbb{Q}).$$

This corollary in particular implies that for any sequence $a_i \in \mathbf{G}(\mathbb{Q})$ with $\deg(a_i) \to \infty$, and for any $\psi \in C_c(\mathbf{G}_S)$,

$$\lim_{i\to\infty} \frac{1}{\deg(a_i)} \sum_{x\in\Gamma a_i\Gamma} \psi(x) = \frac{1}{\tau_S(\Gamma\backslash\mathbf{G}_S)} \int_{\mathbf{G}_S} \psi(g)\, d\tau_S.$$

It is interesting to note that unlike the rational points $\mathbf{G}(\mathbb{Q})$ of bounded height (Theorem 1.6), the Hecke points are equidistributed in $\mathbf{G}_S$ with respect to the invariant measure.

The following corollary presents a stronger version of property (τ) of $\mathbf{G}$ proved by Clozel [15]:

Corollary 3.30. *Let $\mathbf{G}$ be a connected simply connected almost K-simple K-group. Let π denote the quasi-regular representation of $\mathbf{G}(\mathbb{A})$ on $L^2_0(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Let W be a maximal compact subgroup of $\mathbf{G}(\mathbb{A})$. Then there exist an explicit $p = p(\mathbf{G}) < \infty$ such that any W-finite matrix coefficient of π is $L^p(\mathbf{G}(\mathbb{A}))$-integrable.*

4. Volume asymptotics and Construction of $\tilde{\mu}_\iota$

4.1. **Analytic properties of height zeta functions.** Let $\mathbf{G}$ be a connected adjoint semisimple algebraic group over K. Let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be a faithful representation defined over K which has a unique maximal weight. Recall the constants a_ι and b_ι defined in the introduction (1.3): Choosing a maximal torus $\mathbf{T}$ of $\mathbf{G}$ defined over K containing a maximal K-split torus and a set of simple roots Δ of the root system $\Phi(\mathbf{G}, \mathbf{T})$, denote by 2ρ the sum of all positive roots and λ_ι the unique maximal weight of ι.

If

$$2\rho = \sum_{\alpha\in\Delta} u_\alpha \alpha \quad \text{and} \quad \lambda_\iota = \sum_{\alpha\in\Delta} m_\alpha \alpha$$

then

$$a_\iota = \max_{\alpha\in\Delta} \frac{u_\alpha + 1}{m_\alpha} \quad \text{and} \quad b_\iota = \#\{\Gamma_K.\alpha : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\} \tag{4.1}$$

where Γ_K is the absolute Galois group over K.

We fix a height function $\mathrm{H}_\iota = \prod_{v\in R} \mathrm{H}_{\iota,v}$ as defined in (2.6) in the rest of this section. Recall the notation

$$B_T := \{g \in \mathbf{G}(\mathbb{A}) : \mathrm{H}_\iota(g) \le T\}.$$

Given an automorphic character χ, we consider the following functions:

$$\mathcal{Z}_S(s,\chi) := \int_{\mathbf{G}_S} \mathrm{H}_\iota(g)^{-s}\chi(g)\, d\tau_S(g); \qquad \mathcal{Z}^S(s,\chi) := \int_{\mathbf{G}^S} \mathrm{H}_\iota(g)^{-s}\chi(g)\, d\tau^S(g).$$

Lemma 4.2. *There exists $\epsilon > 0$ such that the following hold for any finite $S \subset R$:*

1. *$\mathcal{Z}_S(s,\chi)$ absolutely converges for $\Re(s) \ge a_\iota - \epsilon$.*
2. *$\tau_S(B_T \cap \mathbf{G}_S) = O(T^{a_\iota - \epsilon})$ where the implied constant depends on S.*

Proof. For (1), Recall the Cartan decomposition for each v: $\mathbf{G}(K_v) = U_v A_v^+ \Omega_v U_v$ (2.2). Since the definition of a_ι does not depend on a particular choice of $\mathbf{T}$, we may assume $A_v \subset \mathbf{T}(K_v)$. Choose the set of simple roots $\Delta_v = \{\alpha_1, \cdots, \alpha_r\}$ in the root system $\Phi(\mathbf{G}(K_v), A_v)$ so that the restriction of Δ to A_v is contained in $\Delta_v \cup \{0\}$.

If $2\rho_v$ denotes the sum of all positive roots in $\Phi(\mathbf{G}(K_v), A_v)$ and u_i' denotes the sum of all u_α's for those α such that $\alpha|_{A_v} = 1$, i.e., $u_i' = \sum\{u_\alpha : \alpha|_{A_v} = \alpha_i\}$, then $2\rho_v = 2\rho|_{A_v} = \sum_{i=1}^r u_i'\alpha_i$. Similarly, if $\lambda_{\iota,v} := \lambda_\iota|_{A_v}$ and $m_i' = \sum\{m_\alpha : \alpha|_{A_v} = \alpha_i\}$, we have $\lambda_{\iota,v} = \sum_{i=1}^r m_i'\alpha_i$.

Observe that

$$a_v := \max_{1\le i\le r} \frac{u_i'}{m_i'} \le \max_{\alpha\in\Delta} \frac{u_\alpha}{m_\alpha} < a_\iota.$$

Since λ_ι is the unique maximal weight of ι, we may assume, without loss of generality, that

$$\mathrm{H}_{\iota,v}(kadk') = q_v^{\log_{q_v} |\lambda_\iota(a)|}$$

where $k, k' \in U_v, a \in A_v^+, d \in \Omega_v$ and $q_v = e$ if $v \in R_\infty$.

For $v \in R_\infty$, it is well known (cf. Prop. 5.28 in [35]) that $dg_v = \delta(X)dk_1 dX dk_2$ where for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\delta(X) < C_\epsilon \exp((1+\epsilon)2\rho_v(X)) \quad \text{for all } X \in \log(A_v^+)\ .$$

Hence if $\sigma > 0$,

$$\begin{aligned}
&\int_{\mathbf{G}(K_v)} \mathrm{H}_{\iota,v}(g_v)^{-\sigma} dg_v \le C_\epsilon \int_{\log A_v^+} \exp(-\sigma(\lambda_{\iota,v}(X) + (1+\epsilon)2\rho_v(X)))dX \\
&\le C_\epsilon \prod_{i=1}^r \int_{x_i=0}^\infty \exp(-x_i(\sigma m_i' - u_i'(1+\epsilon)))dx_i.
\end{aligned}$$

Hence the above converges for any $\sigma > a_v$, proving the claim for $v \in R_\infty$.

Let $v \in R_f$. Without loss of generality, we may assume that $\mathrm{H}_{\iota,v}$ is bi-U_v-invariant, and hence

$$\int_{\mathbf{G}(K_v)} \mathrm{H}_{\iota,v}(g_v)^{-\sigma} dg_v = \sum_{ad \in A_v^+\Omega_v} \mathrm{H}_{\iota,v}(ad)^{-\sigma} \tau_v(U_v ad U_v).$$

By [52, Lemma 4.1.1], there exists $c_1 > 0$ such that $\tau_v(U_v ad U_v) \le c_1 \cdot q_v^{2\rho_v(a)}$ for all $ad \in A_v^+\Omega_v$. Hence for some constant $c > 0$,

$$\int_{\mathbf{G}(K_v)} \mathrm{H}_{\iota,v}(g_v)^{-\sigma} dg_v \le c \cdot \sum_{a \in A_v^+} q_v^{-\sigma\lambda_{\iota,v}(a) + 2\rho_v(a)} = c \prod_{i=1}^r \sum_{j=0}^\infty q_v^{-(\sigma m_i' - u_i')j},$$

where the last term converges for any $\sigma > a_v$. Put

$$\epsilon = \frac{1}{2}(a_\iota - \max_{\alpha\in\Delta} \frac{u_\alpha}{m_\alpha})$$

so that $a_\iota - \epsilon > a_v$ for all $v \in R$. The above argument proves the claim (1) for this choice of ϵ. Also, if $v(t) := \tau_S(\{g \in \mathbf{G}_S : \mathrm{H}_{\iota,S}(g) \le t\})$, and $\sigma > a_\iota - \epsilon$,

$$\int_0^\infty t^{-\sigma} dv(t) = \int_{\mathbf{G}_S} \mathrm{H}_{\iota,S}(g)^{-\sigma} d\tau_S(g) < \infty$$

Now the second claim follows from the properties of Laplace/Mellin transform (see, for example, [60, Ch.II,§2]). □

One of the main contribution of the paper by Shalika, Takloo-Bighash and Tschinkel [50] is the regularization of $\mathcal{Z}^S(s,\chi)$ via the Hecke L-functions. Their result stated as [50, Theorem 7.1], together with the results in Tate's thesis on the meromorphic continuation of Hecke L-functions and their boundedness on vertical strips (cf. [11, Prop. 3.16]), implies the following:

Theorem 4.3. *Let S be a finite subset of R and a_ι, b_ι as in (4.1). Then $\mathcal{Z}^S(s,\chi)$ converges absolutely when $\Re(s) > a_\iota$, and there exists $\epsilon > 0$ such that $\mathcal{Z}^S(s,\chi)$ has a meromorphic continuation to $\Re(s) > a_\iota - \epsilon$ with a unique pole at $s = a_\iota$ of order at*

most b_ι. The order of the pole is exactly b_ι for $\chi = 1$. Moreover, for some constants $\kappa \in \mathbb{R}$ and $C > 0$,

$$\left| \frac{(s-a_\iota)^{b_\iota} \mathcal{Z}^S(s,\chi)}{s^{b_\iota}} \right| \le C \cdot (1 + |\operatorname{Im}(s)|)^\kappa$$

for $\Re(s) > a_\iota - \epsilon$.

In [50], it is assumed that $\mathrm{H}_{\iota,v}$ is smooth for $v \in (R - S) \cap R_\infty$ which is stronger than the condition (3) in Definition 2.6. This implies Theorem 4.3 for any S including R_∞. On the other hand, by Lemma 4.2, for any finite $S_1 \subset R$, there exists $\epsilon > 0$ such that the product $\mathcal{Z}_{S_1}(s,\chi) := \int_{\mathbf{G}_{S_1}} \mathrm{H}_\iota(g)^{-s} \chi(g)\, d\tau_{S_1}(g)$ absolutely converges for all $\Re(s) > a_\iota - \epsilon$. Therefore, for any $S_2 \subset S$, the product $\mathcal{Z}^{S_2} = \mathcal{Z}_{S-S_2} \mathcal{Z}^S$ satisfies the properties listed in Theorem 4.3, provided $\mathcal{Z}^S$ does. Therefore Theorem 4.3 holds for any finite $S \subset R$.

We use the following version of Ikehara Tauberian theorem to deduce the volume asymptotics from Theorem 4.3.

Theorem 4.4. *Fix $a > 0$ and $\delta_0 > 0$. Let $\alpha(t)$ be a non-negative non-decreasing function on (δ, ∞) such that*

$$f(s) := \int_{\delta_0}^{\infty} t^{-s}\, d\alpha$$

converges for $\Re(s) > a$. Suppose that for some $\epsilon > 0$,

- *$f(s)$ has a meromorphic continuation to the half plane $\Re(s) > a - \epsilon > 0$ and has a unique pole at $s = a$ with order b;*
- *For some $\kappa \in \mathbb{R}$ and $C > 0$,*

$$\left| \frac{f(s)(s-a)^b}{s^b} \right| \le C \cdot (1 + |\operatorname{Im}(s)|)^\kappa$$

  for $\Re(s) > a - \epsilon$.

Then for some $\delta > 0$,

$$\int_\delta^T d\alpha = \alpha(T) - \alpha(\delta) = \frac{c}{a(b-1)!} \cdot T^a P(\log T) + O(T^{a-\delta}) \quad \text{as } T \to \infty$$

where $c = \lim_{s\to a} (s-a)^b f(s)$ and $P(x)$ is a monic polynomial of degree $b-1$.

Proof. This can be proved by repeating the same argument as in the appendix of [13] simply replacing the sum $\sum_n n^{-s} \alpha_n$ by the integral $\int_{\delta_0}^{\infty} t^{-s}\, d\alpha(t)$. □

4.2. **Definition of $\gamma^S_{W_f}$.** Recall from (2.7) that W_ι denotes the maximal compact subgroup of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant. For any co-finite subgroup W_f of W_ι, recall from (2.5) the definition $G_{W_f} := \ker(\Lambda^{W_f})$ where Λ^{W_f} is the subset of W_f-invariant characters in Λ. We will deduce the asymptotic volume of the intersection $B_T \cap G_{W_f}$, more generally, that of $B_T \cap gG_{W_f} \cap \mathbf{G}^S$ for any $g \in \mathbf{G}(\mathbb{A})$ and any finite $S \subset R$. In this subsection, we will define a function $\gamma^S_{W_f} : \mathbf{G}(\mathbb{A}) \to \mathbb{R}_{>0}$ which appears in the main asymptotic of these volumes.

Definition 4.5. *For a finite $S \subset R$ and a co-finite subgroup W_f of W_ι, define a function $\gamma^S_{W_f} : \mathbf{G}(\mathbb{A}) \to \mathbb{R}_{>0}$ by*

$$\gamma^S_{W_f}(g) := \sum_{\chi \in \Lambda^{W_f}} c^S_\chi \cdot \chi(g) \quad \textit{with} \quad c^S_\chi := \lim_{s \to a_\iota} (s - a_\iota)^{b_\iota} \mathcal{Z}^S(s, \chi).$$

For simplicity, when $S = \emptyset$, we set $\gamma_{W_f} = \gamma^\emptyset_{W_f}$ and $c_\chi = c^\emptyset_\chi$.

By Theorem 4.3, the limits appearing in the definition of $\gamma^S_{W_f}$ exist. To show $\gamma^S_{W_f}$ is well-defined, it remains to show the following:

Proposition 4.6. *For any $S \subset R$ and any co-finite subgroup W_f of W_ι, $\gamma^S_{W_f}(g) > 0$ for any $g \in \mathbf{G}(\mathbb{A})$.*

We need some preliminaries to prove this proposition.

Lemma 4.7. *The following statements hold for any compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$:*

(1) *If $\mathbf{G}^\circ_\infty$ denotes the identity component of $\mathbf{G}_\infty$,*
$$\mathbf{G}(K)\mathbf{G}^\circ_\infty W_f \subset G_{W_f}.$$
(2) $\#\Lambda^{W_f} = [\mathbf{G}(\mathbb{A}) : G_{W_f}] < \infty$.
(3) *For any $g \in \mathbf{G}(\mathbb{A})$,*
$$\sum_{\chi \in \Lambda^{W_f}} \chi(g) = \begin{cases} \#\Lambda^{W_f} & \textit{if } g \in G_{W_f} \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Since $\mathbf{G}^\circ_\infty$ is a connected semisimple group, $\mathbf{G}^\circ_\infty \subset \ker(\chi)$ for any $\chi \in \Lambda$. On the other hand, $\chi(\mathbf{G}(K)) = 1$ for any $\chi \in \Lambda$, by the definition of an automorphic character. Hence (1) follows. Since $\mathbf{G}^\circ_\infty$ has a finite index in $\mathbf{G}_\infty$, it follows from [44, Theorem 5.1] that there exist finitely many $u_1, \cdots, u_h \in \mathbf{G}(\mathbb{A})$ such that

$$\mathbf{G}(\mathbb{A}) = \cup_{i=1}^h \mathbf{G}(K) u_i \mathbf{G}^\circ_\infty W_f.$$

It follows by (1) that $[\mathbf{G}(\mathbb{A}) : G_{W_f}] < \infty$. Now the quotient $G_{W_f} \backslash \mathbf{G}(\mathbb{A})$ is a finite abelian group. In particular, G_{W_f} is an open subgroup of $\mathbf{G}(\mathbb{A})$ and hence any character of the group $G_{W_f} \backslash \mathbf{G}(\mathbb{A})$ can be considered as a continuous character of

$\mathbf{G}(\mathbb{A})$ which is trivial on G_{W_f}, that is, an element of Λ^{W_f}. Conversely, any element of Λ^{W_f} defines a character of $G_{W_f}\backslash\mathbf{G}(\mathbb{A})$.

Now consider the scalar product on the space functions of $G_{W_f}\backslash\mathbf{G}(\mathbb{A})$ given by

$$\langle\psi_1|\psi_2\rangle := \frac{1}{[\mathbf{G}(\mathbb{A}):G_{W_f}]}\sum_{x\in G_{W_f}\backslash\mathbf{G}(\mathbb{A})}\psi_1(x)\overline{\psi_2(x)}.$$

We claim that Λ^{W_f} forms an orthonormal set with respect to this scalar product. For $\chi,\chi'\in\Lambda^{W_f}$, observe that

$$\begin{aligned}
&\sum_{x\in G_{W_f}\backslash\mathbf{G}(\mathbb{A})}\chi(x)\overline{\chi'(x)}\\
&=\tau(\mathbf{G}(K)\backslash G_{W_f})^{-1}\int_{g\in\mathbf{G}(K)\backslash G_{W_f}}\left(\sum_{x\in G_{W_f}\backslash\mathbf{G}(\mathbb{A})}\chi(gx)\overline{\chi'(gx)}\right)d\tau(g)\\
&=\tau(\mathbf{G}(K)\backslash G_{W_f})^{-1}\int_{\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})}\chi(x)\overline{\chi'(x)}\,d\tau(x).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\langle\chi|\chi'\rangle &= \frac{1}{[\mathbf{G}(\mathbb{A}):G_{W_f}]}\tau(\mathbf{G}(K)\backslash G_{W_f})^{-1}\int_{\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})}\chi(x)\overline{\chi'(x)}\,d\tau(x)\\
&=\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^{-1}\langle\chi,\chi'\rangle_{L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))}\\
&=\langle\chi,\chi'\rangle_{L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))}
\end{aligned}$$

since $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))=1$. Since Λ is an orthonormal subset of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, the claim follows. Now (2) and (3) follow from the duality of finite groups. $\square$

Lemma 4.8. *Fix a finite subset $S\subset R$. Let $U\subset\mathbf{G}^S$ be an open subset such that $\mathbf{G}^S=FU$ for some finite subset F of $\mathbf{G}^S$. Then for ($\sigma\in\mathbb{R}$)*

$$\liminf_{\sigma\to a_\iota}(\sigma-a_\iota)^{b_\iota}\int_U\mathbf{H}_\iota(h)^{-\sigma}d\tau^S(h)>0.$$

Proof. By Theorem 4.3,

$$c_0:=\lim_{s\to a_\iota}(s-a_\iota)^{b_\iota}\int_{\mathbf{G}^S}\mathbf{H}_\iota(h)^{-s}d\tau^S(h)$$

exists and is non-zero. It is then clear that $c_0>0$ since $\mathbf{H}_\iota$ is a positive function on $\mathbf{G}^S$. For any $f\in F$, we can find $c_f\geq 1$ such that for all $h\in\mathbf{G}^S$, $c_f^{-1}\mathbf{H}_\iota(h)\leq\mathbf{H}_\iota(fh)\leq c_f\mathbf{H}_\iota(h)$. Without loss of generality, we may assume $\mathbf{H}_\iota(h)\geq 1$ for all $h\in\mathbf{G}^S$. Now for $\sigma<a_\iota+1$, we have

$$\int_{\mathbf{G}^S}\mathbf{H}_\iota(h)^{-\sigma}d\tau^S(h)\leq(\max_{f\in F}c_f)^{a_\iota+1}\cdot\int_U\mathbf{H}_\iota(h)^{-\sigma}d\tau^S(h).$$

Hence the claim follows. $\square$

Lemma 4.9. *For any finite $S \subset R$ and a co-finite subgroup W_f of W_ι, there is a map $g \mapsto s_g : \mathbf{G}_S \to \mathbf{G}^S$ which factors through $(\mathbf{G}_S \cap G_{W_f})\backslash \mathbf{G}_S$ for which the following hold:*

(1)
$$G_{W_f} = \bigcup_{g \in (G_{W_f} \cap \mathbf{G}_S)\backslash \mathbf{G}_S} (G_{W_f} \cap \mathbf{G}_S) g\, s_g (G_{W_f} \cap \mathbf{G}^S);$$

(2) *for any $\varphi \in C_c(G_{W_f})$,*
$$\int_{G_{W_f}} \varphi\, d\tau_{W_f} = \int_{g \in \mathbf{G}_S} \int_{h \in \mathbf{G}^S \cap G_{W_f}} \varphi(g s_g h)\ d\tau^S(h)\, d\tau_S(g)$$
if τ^S and τ_S are normalized so that $\tau_{W_f} = \tau^S \times \tau_S$ locally.

Proof. Let pr denote the restriction of the projection map $\mathbf{G}(\mathbb{A}) \to \mathbf{G}_S$ to G_{W_f}. Since $\mathbf{G}(K)$ is dense in $\mathbf{G}_S$ by the weak approximation and the image $\mathrm{pr}(G_{W_f})$ is an open subgroup containing $\mathbf{G}(K)$, the map pr is surjective.

Note that $(G_{W_f} \cap \mathbf{G}^S)(G_{W_f} \cap \mathbf{G}_S)$ is a normal subgroup of G_{W_f}, and that the map pr induces an isomorphism, say $\tilde{\mathrm{pr}}$, between $(G_{W_f} \cap \mathbf{G}_S)(G_{W_f} \cap \mathbf{G}^S)\backslash G_{W_f}$ and $(G_{W_f} \cap \mathbf{G}_S)\backslash \mathbf{G}_S$. For each $g \in (G_{W_f} \cap \mathbf{G}_S)\backslash \mathbf{G}_S$, choose $s_g \in \mathbf{G}^S \cap \tilde{\mathrm{pr}}^{-1}(g)$. This yields the decomposition
$$G_{W_f} = \cup_{g \in (G_{W_f} \cap \mathbf{G}_S)\backslash \mathbf{G}_S} (G_{W_f} \cap \mathbf{G}^S)(G_{W_f} \cap \mathbf{G}_S) s_g\, g.$$
Since $(G_{W_f} \cap \mathbf{G}_S) s_g\, g = g\, s_g (G_{W_f} \cap \mathbf{G}_S)$, (1) follows. It is easy to deduce (2) from (1). $\square$

Proof of Proposition 4.6: For $g \in \mathbf{G}^S$, consider

$$\sum_{\chi \in \Lambda^{W_f}} \mathcal{Z}^S(s,\chi)\chi(g) = \sum_{\chi \in \Lambda^{W_f}} \int_{\mathbf{G}^S} \mathrm{H}_\iota(h)^{-s} \chi(gh)\, d\tau^S(h) \tag{4.10}$$
$$= (\#\Lambda^{W_f}) \int_{g^{-1}G_{W_f} \cap \mathbf{G}^S} \mathrm{H}_\iota(h)^{-s}\, d\tau^S(h)$$

where the second equality holds by Lemma 4.7(3).

Hence
$$(\#\Lambda^{W_f})^{-1} \cdot \gamma^S_{W_f}(g) = \lim_{s \to a_\iota} (s - a_\iota)^{b_\iota} \int_{g^{-1}G_{W_f} \cap \mathbf{G}^S} \mathrm{H}_\iota(h)^{-s}\, d\tau^S(h),$$

which is equal to, along $\sigma \in \mathbb{R}$,
$$\liminf_{\sigma \to a_\iota} (\sigma - a_\iota)^{b_\iota} \int_{g^{-1}G_{W_f} \cap \mathbf{G}^S} \mathrm{H}_\iota(h)^{-\sigma}\, d\tau^S(h).$$

Therefore by Lemma 4.8, $\gamma^S_{W_f}(g) > 0$ for $g \in \mathbf{G}^S$. Since $\mathbf{G}(\mathbb{A}) = \mathbf{G}_S\mathbf{G}^S$, it suffices to prove $\gamma^S_{W_f}(hx) > 0$ for any $h \in \mathbf{G}_S$ and $x \in \mathbf{G}^S$. Let $s_h \in \mathbf{G}^S$ be as defined in Lemma 4.9. By (1) of the same lemma,

$$\chi(h) = \chi(s_h^{-1})$$

for any $\chi \in \Lambda^{W_f}$. This implies that for any $x \in \mathbf{G}^S$, $\gamma^S_{W_f}(hx) = \gamma^S_{W_f}(s_h^{-1}x)$. Since $s_h^{-1}x \in \mathbf{G}^S$, by Theorem 4.13, $\gamma^S_{W_f}(s_h^{-1}x) > 0$ and hence $\gamma^S_{W_f}(hx) > 0$. This finishes the proof.

The functions $\gamma^S_{W_f}$ are related for different S's by the following:

Proposition 4.11. *Let $S \subset S'$ be finite subsets of R and W_f a co-finite subgroup of W_ι,*

(1) *for any $g \in \mathbf{G}^S$,*

$$\gamma^S_{W_f}(g) = \int_{h\in\mathbf{G}_{S_0}} \mathrm{H}^{-a_\iota}_{\iota,S_0}(h)\, \gamma^{S'}_{W_f}(hg)\, d\tau_{S_0}, \quad \textit{for } S_0 = S' - S.$$

(2) *In particular, for any finite $S \subset R$,*

$$\gamma_{W_f}(e) = \int_{\mathbf{G}_S} \mathrm{H}^{-a_\iota}_{\iota,S}\ \gamma^S_{W_f}\ d\tau_S.$$

(3) *$\gamma_{W_f} = \sum_{\chi\in\Lambda} c_\chi\chi$; in particular, $\gamma_{W_f} = \gamma_{W_\iota}$.*

Proof. Note that $\mathbf{G}^S = \mathbf{G}_{S_0}\mathbf{G}^{S'}$ and $\tau^S = \tau_{S_0} \times \tau^{S'}$. Since $\int_{\mathbf{G}_{S_0}} \mathrm{H}^{-a_\iota}_{\iota,S_0}\ \chi\ d\tau_{S_0}$ exists for any $\chi \in \Lambda$ by Lemma 4.2, we deduce

$$\begin{aligned}
\gamma^S_{W_f}(g) &= \sum_{\chi\in\Lambda^{W_f}} \left(\int_{\mathbf{G}_{S_0}} \mathrm{H}^{-a_\iota}_{\iota,S_0}\, \chi\, d\tau_S\right) \cdot \left(\lim_{s\to a_\iota^+} (s-a_\iota)^{b_\iota}\, \mathcal{Z}^{S'}(s,\chi)\right) \cdot \chi(g) \\
&= \int_{h\in\mathbf{G}_{S_0}} \mathrm{H}_{\iota,S_0}(h)^{-a_\iota} \ \left(\sum_{\chi\in\Lambda^{W_f}} c^{S'}_\chi\ \chi(gh)\right)\ d\tau_{S_0}(h) \\
&= \int_{h\in\mathbf{G}_{S_0}} \mathrm{H}_{\iota,S_0}(h)^{-a_\iota} \gamma^{S'}_{W_f}(gh)\, d\tau_{S_0}(h).
\end{aligned}$$

Hence (1) follows. By putting $S = \emptyset$, (2) follows.

Since H_ι is W_f-invariant, it follows from Lemma 4.12 below $c_\chi = 0$ for any $\chi \in \Lambda - \Lambda^{W_f}$. Hence the claim holds. $\square$

Lemma 4.12. *Let $Y = \mathbf{G}(\mathbb{A})$ or $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$. Let W_f be a co-finite subgroup of W_ι and let $\chi \in \Lambda - \Lambda^{W_f}$. Then for any W_f-invariant function ψ on Y, we have*

$$\int_Y \chi(g)\psi(g)\, d\tau(g) = 0,$$

if the integral exists. In particular, if the support of ψ is contained in $\mathbf{G}(K)\backslash G_{W_f}$ and $\int_{\mathbf{G}(K)\backslash G_{W_f}} \psi \, d\tau = 0$, then for any $\chi \in \Lambda$,

$$\int_{\mathbf{G}(K)\backslash G_{W_f}} \chi \cdot \psi \, d\tau = 0.$$

Proof. Since $\chi \in \Lambda - \Lambda^{W_f}$ there exists $w \in W_f$ such that $\chi(w) \neq 1$. Since ψ is W_f-invariant and τ is invariant,

$$\begin{aligned} &\int_{\mathbf{G}(\mathbb{A})} \chi(g)\psi(g) \, d\tau_0(g) \\ &= \int_{\mathbf{G}(\mathbb{A})} \chi(wg)\psi(wg) \, d\tau(g) \\ &= \chi(w) \int_{\mathbf{G}(\mathbb{A})} \chi(g)\psi(g) \, d\tau(g). \end{aligned}$$

This equality implies the first claim immediately. For the second claim, it suffices to note that for $\chi \in \Lambda^{W_f}$, $\chi = 1$ on G_{W_f}. □

4.3. **Volume asymptotic.**

Theorem 4.13. *Let $a_\iota \in \mathbb{Q}^+$ and $b_\iota \in \mathbb{N}$ be as in (4.1). Then for any finite subset $S \subset R$, any co-finite subgroup W_f of W_ι and $g \in \mathbf{G}^S$, there exists a monic polynomial $P(x)$ of degree $b_\iota - 1$ and a positive real number δ such that*

$$\tau^S(B_T \cap gG_{W_f} \cap \mathbf{G}^S) = \frac{\gamma^S_{W_f}(g^{-1})}{\#\Lambda^{W_f} \cdot a_\iota (b_\iota - 1)!} \cdot T^{a_\iota} P(\log T) + O(T^{a_\iota - \delta}). \tag{4.14}$$

In particular, as $T \to \infty$,

$$\tau^S(B_T \cap gG_{W_f} \cap \mathbf{G}^S) \sim \frac{\gamma^S_{W_f}(g^{-1})}{\#\Lambda^{W_f} \cdot a_\iota (b_\iota - 1)!} \cdot T^{a_\iota} (\log T)^{b_\iota - 1}. \tag{4.15}$$

Proof. By Lemma 2.5, B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$ and hence $\tau^S(B_T \cap \mathbf{G}^S) < \infty$ for each $T \geq 1$ and for any finite S. By the same lemma, $\delta_0 := \inf_{g \in \mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g) > 0$. Define

$$\alpha(t) = \tau^S(B_t \cap gG_{W_f} \cap \mathbf{G}^S) \quad \text{for } t \in [\delta_0, \infty),$$

and

$$f(s) = \int_{\delta_0}^{\infty} t^{-s} \, d\alpha.$$

Then by (4.10),

$$f(s) = \int_{gG_{W_f} \cap \mathbf{G}^S} \mathrm{H}_\iota(h)^{-s} \, d\tau^S(h) = (\#\Lambda^{W_f})^{-1} \sum_{\chi \in \Lambda^{W_f}} \mathcal{Z}^S(s, \chi)\chi(g^{-1}).$$

Since

$$\lim_{s\to a_\iota}(s-a_\iota)^{b_\iota}f(s)=(\#\Lambda^{W_f})^{-1}\gamma^S_{W_f}(g)>0$$

by Proposition 4.6, the claim follows from Theorems 4.3 and 4.4. □

Lastly in this subsection, we show that the volume asymptotic for $\tau_{W_f}(B_T\cap G_{W_f})$ is independent of W_f's.

Corollary 4.16. *For $g\in \mathbf{G}(\mathbb{A})$ and any co-finite subgroup W_f of W_ι,*

$$\tau(B_T\cap gG_{W_\iota})\sim_T [G_{W_\iota}:G_{W_f}]\cdot\tau(B_T\cap gG_{W_f}).$$

In particular,

$$\lim_{T\to\infty}\frac{\tau_{W_f}(B_T\cap G_{W_f})}{\tau_{W_\iota}(B_T\cap G_{W_\iota})}=1.$$

Proof. Since $\gamma_{W_f}(g)=\gamma_{W_\iota}(g)$ by Proposition 4.11, the first claim follows from Theorem 4.13 and Lemma 4.7(2). Since the restriction of τ to G_{W_f} is equal to $[\mathbf{G}(\mathbb{A}):G_{W_f}]\cdot\tau_{W_f}$, the second claim follows from the first one. □

4.4. **Construction of $\tilde\mu_\iota$.** Let $\bar\iota:\mathbf{G}\to\mathbb{P}(\mathrm{M_N})$ denote the projective embedding obtained by the composition of ι with the canonical projection from $\mathrm{GL}_N\to\mathbb{P}(\mathrm{M_N})$. For each $v\in R$, denote by $X_{\iota,v}$ the closure of $\bar\iota(\mathbf{G}(K_v))$ in $\mathbb{P}(\mathrm{M}_N(K_v))$, and set

$$X_\iota=\prod_{v\in R}X_{\iota,v}.$$

Fix a height function $\mathrm{H}_\iota=\prod_{v\in R}\mathrm{H}_{\iota,v}$ on the associated adele group $\mathbf{G}(\mathbb{A})$ relative to ι as in Definition 2.6. For finite subset $S\subset R$ and set $\mathrm{H}_{\iota,S}=\prod_{v\in S}\mathrm{H}_{\iota,v}$ and $X_{\iota,S}=\prod_{v\in S}X_{\iota,v}$. Without loss of generality, we may consider $\mathbf{G}(K_v)$ as a subset of $X_{\iota,v}$. We will first construct a family measures $\{\mu_{\iota,W_f,S}\}$ on $X_{\iota,S}$ for all finite S and for all co-finite subgroups W_f of W_ι, and put them together to define the measure $\tilde\mu_\iota$ on X_ι.

Recall the definition of the function $\gamma^S_{W_f}$ on $\mathbf{G}(\mathbb{A})$ from (4.5), and the notation $\gamma_{W_\iota}(e)=\gamma^{\emptyset}_{W_\iota}(e)$. By Lemma 4.2 and Propositions 4.6, 4.11 (2), the following is a well-defined probability measure on $\mathbf{G}_S$ (which will be in fact considered as a measure on $X_{\iota,S}$):

$$d\mu_{\iota,W_f,S}(g):=\gamma_{W_\iota}(e)^{-1}\cdot\mathrm{H}_{\iota,S}(g)^{-a_\iota}\cdot\gamma^S_{W_f}(g)\,d\tau_S(g).$$

Remark 4.17. Let $\mathbf{G}'_S$ denote the derived subgroup of $\mathbf{G}_S$. Then $[\mathbf{G}_S:\mathbf{G}'_S]<\infty$ [44, Proposition 3.17]. Then for any $\psi\in C(X_{\iota,S})$, since the projection of $\gamma^S_{W_f}$ to $\mathbf{G}_S$ factors through $\mathbf{G}'_S$, we deduce that

$$\mu_{\iota,W_f,S}(\psi)=\gamma_{W_\iota}(e)^{-1}\sum_{u\in\mathbf{G}_S/\mathbf{G}'_S}\gamma^S_{W_f}(u)\cdot\int_{u\mathbf{G}'_S}\mathrm{H}_{\iota,S}(g)^{-a_\iota}\psi(g)\,d\tau_S(g).$$

Since $\gamma_{W_f}^S(u) > 0$ for each u, it follows that the measure $\mu_{\iota,W_f,S}$ is equivalent to a Haar measure on $\mathbf{G}_S$, considered as a measure on $X_{\iota,S}$.

For $W_f < W_\iota$, denote by $C(X_\iota)^{W_f}$ the closed subspace of $C(X_\iota)$ consisting of functions which are (right)-invariant under W_f.

Theorem 4.18. *There exists a unique probability measure $\tilde{\mu}_\iota$ on X_ι such that for any $\psi \in \bigcup_{W_f<W_\iota \text{ co-finite}} C(X_\iota)^{W_f}$,*

$$\tilde{\mu}_\iota(\psi) = \gamma_{W_\iota}(e)^{-1} \cdot \sum_{\chi\in\Lambda} \lim_{s\to a_\iota^+} (s-a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s}\chi(g)\, \psi(g) d\tau(g). \tag{4.19}$$

Proof. Define a linear functional μ_{ι,W_f} on $C(X_\iota)^{W_f}$ by

$$\mu_{\iota,W_f}(\psi) = \gamma_{W_\iota}(e)^{-1} \cdot \sum_{\chi\in\Lambda^{W_f}} \lim_{s\to a_\iota^+} (s-a_\iota)^{b_\iota} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g)^{-s}\chi(g)\, \psi(g) d\tau(g). \tag{4.20}$$

We first claim that μ_{ι,W_f} is well-defined, positive and bounded by 1 and $\mu_{\iota,W_f}(1) = 1$. For each finite set of places S, let $C(W_f,S)$ denote the subset of $C(X_\iota)^{W_f}$ consisting of functions which factor through $X_{\iota,S}$. The restriction of $\psi \in C(W_f,S)$ to $X_{\iota,S}$ will also be denoted by ψ by abuse of notation. By Proposition 4.11 (1), the measures $\mu_{\iota,W_f,S}$ are compatible in the sense that for $S \subset S'$, the restriction of $\mu_{\iota,W_f,S'}$ to $C(W_f,S)$ coincides with $\mu_{\iota,W_f,S}$. Observe that for $\psi \in C(W_f,S)$,

$$\mu_{\iota,W_f}(\psi) = \mu_{\iota,W_f,S}(\psi). \tag{4.21}$$

Hence the limit exists in (4.20) for all $\psi \in C(W_f,S)$ for each finite S. Since $\bigcup_S C(W_f,S)$ is dense in $C(X_\iota)^{W_f}$ where S ranges over all finite subsets of R and μ_{ι,W_f} is a linear functional with

$$|\mu_{\iota,W_f}(\psi)| \le \|\psi\|_\infty \quad \text{for any } \psi \in C(X_\iota)^{W_f}$$

it follows that the limit exists in (4.20) for any $\psi \in C_c(X_\iota)^{W_f}$ and hence μ_{ι,W_f} is well-defined. The other claims on μ_{ι,W_f} are now clear.

By applying Lemma 4.12 for $\psi = \mathrm{H}_\iota^{-s}$, the family μ_{ι,W_f} of linear functionals on $C(X_\iota)^{W_f}$ is compatible, in the sense that if $V_f \subset W_f$ are co-finite subgroups of W_ι, then

$$\mu_{\iota,V_f}|_{C(X_\iota)^{W_f}} = \mu_{\iota,W_f}.$$

Hence (4.19) is well-defined on $\mathcal{C}_0 := \bigcup_{W_f<W_\iota \text{ co-finite}} C(X_\iota)^{W_f}$.

Since $\mathcal{C}_0$ is dense in $C(X_\iota)$, there exists a unique positive linear functional $\tilde{\mu}_\iota$ on $C(X_\iota)$ satisfying (4.19). □

Proposition 4.22. *For any finite $S \subset R$, the projection $\tilde{\mu}_{\iota,S}$ of $\tilde{\mu}_\iota$ on $X_{\iota,S}$ is equivalent to a Haar measure of $\mathbf{G}_S$.*

Proof. Recall that $\mathbf{G}'_S$ denotes the derived subgroup of $\mathbf{G}_S$. We first claim that if V_f, W_f are co-finite subgroups of $(W_\iota \cap \mathbf{G}'_S) \times (W_\iota \cap \mathbf{G}^S)$, then $\mu_{\iota,W_f,S} = \mu_{\iota,V_f,S}$. Indeed, by definition of $\mu_{\iota,W_f,S}$ and $\gamma^S_{W_f}$, it is sufficient to show that for all $\chi \notin \Lambda^{W_f}$, $c^S_\chi = 0$, and so by symmetry:

$$\gamma^S_{V_f}(g) = \sum_{\chi \in \Lambda^{V_f} \cap \Lambda^{W_f}} c^S_\chi \cdot \chi(g) = \gamma^S_{W_f}(g).$$

Let $\chi \notin \Lambda^{W_f}$, so that $\chi(w) \neq 1$ for some $w \in W_f$. Write $w = w_S w^S$, where $w_S \in \mathbf{G}'_S$ and $w^S \in \mathbf{G}^S$. Then $\chi(w^S) = \chi(w) \neq 1$ since $w_S \in \mathbf{G}'_S$, and hence

$$c^S_\chi = \lim_{s \to a_\iota} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}^S} \mathrm{H}_\iota(g)^{-s} \chi(g) d\tau^S(g).$$

Since $W_f \subset W_\iota$, and $w^S \in W_\iota$,

$$\int_{\mathbf{G}^S} \mathrm{H}_\iota(g)^{-s} \chi(g) d\tau^S(g) = \int_{\mathbf{G}^S} \mathrm{H}_\iota(gw^S)^{-s} \chi(gw^S) d\tau^S(g)$$
$$= \chi(w^S) \int_{\mathbf{G}^S} \mathrm{H}_\iota(g)^{-s} \chi(g) d\tau^S(g),$$

which proves that this integral is zero for all s, and so $c_{s,\chi} = 0$.

Define $\tilde{\mu}_{\iota,S} = \mu_{\iota,W_f,S}$ for any $W_f \subset (W_\iota \cap \mathbf{G}'_S) \times (W_\iota \cap \mathbf{G}^S)$. This measure is absolutely continuous, by the remark 4.17 on $\mu_{\iota,W_f,S}$. We claim that $\tilde{\mu}_{\iota,S}$ is precisely the projection of $\tilde{\mu}_\iota$. For $\psi \in C(W_f, S)$ for some $W_f < (W_\iota \cap \mathbf{G}'_S) \times (W_\iota \cap \mathbf{G}^S)$, we have

$$\tilde{\mu}_\iota(\psi) = \mu_{\iota,W_f}(\psi) = \mu_{\iota,W_f,S}(\psi) = \tilde{\mu}_{\iota,S}(\psi).$$

Since the union $\cup_{W_f} C(W_f, S)$, where W_f ranges over co-finite subgroups of $\mathbf{G}(\mathbb{A}_f)$ contained in $(W_\iota \cap \mathbf{G}'_S)(W_\iota \cap \mathbf{G}^S)$, is dense in $C(X_{\iota,S})$, this finishes the proof. □

Let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be an absolutely irreducible representation defined over K with the highest weight λ_ι. Set

$$\Delta_\iota = \{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}.$$

For $\alpha \in \Delta$, we denote by $\check{\alpha}$ the corresponding coroot. It follows from Theorem 7.1 in [50] that if for a finite subset $S \subset R$ and an automorphic character χ,

$$c^S_\chi = \lim_{s \to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}^S} \mathrm{H}_\iota(g)^{-s} \chi(g)\, dg \neq 0,$$

then

$$\chi(\check{\alpha}) = 1 \quad \text{for all } \alpha \in \Delta_\iota, \tag{4.23}$$

and conversely if (4.23) holds, then $c^S_\chi \neq 0$ for all sufficiently large $S \subset R$.

Remark 4.24.

We discuss some examples to illuminate the properties of the measure $\tilde{\mu}_\iota$.

(1) Suppose that λ_ι is a multiple of $2\rho + \sum_{\alpha\in\Delta}\alpha$. In particular, this holds for λ_ι corresponding to the anticanonical class and for all rank 1 groups.

Then $\Delta_\iota = \Delta$. If a character $\chi \in \Lambda$ satisfies (4.23) then it follows from the Cartan decomposition (2.1) that $\chi(\mathbf{G}^S) = 1$ for sufficiently large S and by the weak approximation, $\chi = 1$. This shows that $c_\chi^S = 0$ for every finite $S \subset R$ and every $\chi \in \Lambda^{W_f} - \{1\}$, so $\gamma_{W_f}^S$ is equal to $\lim_{s\to a_\iota}(s-a_\iota)^{b_\iota}\mathcal{Z}^S(s,1)$. Hence by Theorem 4.13,

$$\#\mathbf{G}(K) \cap B_T \sim_T \tau(B_T); \tag{4.25}$$

and

$$\tilde{\mu}_\iota = \prod_{v\in R} \frac{\mathrm{H}_{\iota,v}(g_v)^{-a_\iota}\, d\tau_v(g_v)}{\int_{\mathbf{G}(K_v)} \mathrm{H}_{\iota,v}(g_v)^{-a_\iota}\, d\tau_v(g_v)}. \tag{4.26}$$

(2) Suppose that K has class number one, $\mathbf{G}$ is K-split and $W_\iota = \prod_{v\in R_f}\mathbf{G}(\mathcal{O}_v)$ (with respect to the canonical model over the ring $\mathcal{O}$ of integers).

According to Remark in Section 2 in [28],

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(K)\mathbf{G}_\infty^\circ W_\iota.$$

Hence, $\Lambda^{W_\iota} = \{1\}$, and consequently, (4.25) holds and μ_{ι,W_ι} is given by (4.26).

(3) (cf. Example 8.10, [50]) Let $\mathbf{G} = \mathrm{PGL}_4$ and ι be the adjoint representation. By [44, §8.2], there exists a lattice $L \subset \mathfrak{pgl}_4(K)$ (i.e., an $\mathcal{O}$-module of full rank) such that $\mathbf{G}$ has class number 2 with respect to L. We take the height function $\mathrm{H} = \prod_{v\in R}\mathrm{H}_v$ where H_v is the maximum norm with respect to L for $v \in R_f$. The group W_ι is given by $\prod_{v\in R_f}\mathrm{Stab}_{\mathbf{G}(K_v)}(L\otimes\mathcal{O}_v)$. By [44, §8.2], $\mathbf{G}(K)\mathbf{G}_\infty W_\iota$ is a normal subgroup of index 2 in $\mathbf{G}(\mathbb{A})$. If we additionally assume that the number field K is totally complex, then $\mathbf{G}_\infty$ is connected and, hence, $\Lambda^{W_\iota} = \{1,\chi\}$ for some automorphic character χ of order 2. Every automorphic character of $\mathbf{G}(\mathbb{A})$ is of the form $\eta\circ\det$ where η is a Hecke character such that $\eta^4 = 1$. Since the map $\det : \mathrm{PGL}_4(K_v) \to K_v^\times/(K_v^\times)^4$ is surjective for every $v \in R$, it follows that $\chi = \eta\circ\det$ with $\eta^2 = 1$. In this case, the roots and coroots are given by

$$\alpha_i(\mathrm{diag}(a_1,\ldots,a_4)) = a_i a_{i+1}^{-1}, \quad \check{\alpha}_i(t) = \mathrm{diag}(\underbrace{t,\ldots,t}_{i},1,\ldots,1)$$

for $i = 1,2,3$, and

$$\lambda_\iota = \alpha_1+\alpha_2+\alpha_3, \quad 2\rho = 3\alpha_1+4\alpha_2+3\alpha_3.$$

Hence, $a_\iota = 5$, $b_\iota = 1$, $\Delta_\iota = \{\alpha_2\}$. Then (4.23) is equivalent to $\eta^2 = 1$, and we deduce that $c_\chi^S \neq 0$ for sufficiently large finite $S \subset R$. Since the function

$\gamma^S_{W_\iota} = c_1^S + c_\chi^S\chi$ restricted to $\mathbf{G}_S$ is not constant for sufficiently large $S \subset R$, we conclude that

$$\mu_{\iota,W_\iota,S} \neq \prod_{v\in S} \frac{\mathrm{H}_{\iota,v}(g_v)^{-a_\iota}\, d\tau_v(g_v)}{\int_{\mathbf{G}(K_v)} \mathrm{H}_{\iota,v}(g_v)^{-a_\iota}\, d\tau_v(g_v)}.$$

We also note that in this case, Theorem 4.13 implies that for an automorphic character χ such that $c_\chi \neq 0$, we have

$$\lim_{T\to\infty} \frac{\tau(B_T \cap G_{W_\iota})}{\tau(B_T)} = c_1^{-1}\cdot \frac{1}{2}(c_1 + c_\chi) \neq \frac{1}{2}.$$

In particular, it may happen that in Theorem 1.10, $\tau(B_T)$ is not asymptotic to $[\mathbf{G}(\mathbb{A}) : G_{W_\iota}] \cdot \tau(B_T \cap G_{W_\iota})$ as $T \to \infty$.

4.5. Equidistribution of height balls $B_T \cap G_{W_f}$ with respect to $\tilde{\mu}_\iota$.

Proposition 4.27. *Let $W_f < W_\iota$ be a co-finite subgroup. Then for any $\psi \in C(X_\iota)^{W_f}$,*

$$\lim_{T\to\infty} \frac{1}{\tau_{W_f}(B_T \cap G_{W_f})} \int_{B_T\cap G_{W_f}} \psi\, d\tau_{W_f} = \tilde{\mu}_\iota(\psi).$$

Proof. To prove the proposition, we may assume $\psi \in C(W_f, S)$ for some finite S, since these functions form a dense subset of $C(X_\iota)^{W_f}$. Let χ_{B_T} denote the characteristic function of the set B_T. By Lemma 4.9 (2), we have a map $g \in \mathbf{G}_S \mapsto s_g \in \mathbf{G}^S$ such that

$$\int_{B_T\cap G_{W_f}} \psi\, d\tau_{W_f} = \int_{g\in \mathbf{G}_S} \psi(g) \int_{h\in \mathbf{G}^S \cap G_{W_f}} \chi_{B_T}(g s_g h)\ d\tau^S(h)\, d\tau_S(g).$$

Since $\mathrm{H}_\iota(g s_g h) = \mathrm{H}_{\iota,S}(g)\,\mathrm{H}_\iota(s_g h)$,

$$\begin{aligned}\int_{h\in \mathbf{G}^S\cap G_{W_f}} \chi_{B_T}(g s_g h)\ d\tau^S(h) &= \tau^S\{h \in \mathbf{G}^S \cap G_{W_f} : \mathrm{H}_\iota(s_g h) < T\,\mathrm{H}_{\iota,S}(g)^{-1}\}\\ &= \tau^S(s_g^{-1} B_{T\cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap G_{W_f} \cap \mathbf{G}^S)\\ &= \tau^S(B_{T\cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap s_g G_{W_f} \cap \mathbf{G}^S)\end{aligned}$$

Hence

$$\int_{B_T\cap G_{W_f}} \psi\, d\tau_{W_f} = \int_{g\in\mathbf{G}_S} \psi(g)\tau^S(B_{T\cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap s_g G_{W_f} \cap \mathbf{G}^S) d\tau_S(g). \tag{4.28}$$

Setting

$$y_T(g) := \frac{\tau^S(B_{T\cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap s_g G_{W_f} \cap \mathbf{G}^S)}{\tau^S(B_T \cap G_{W_f} \cap \mathbf{G}^S)},$$

we claim that for some constant $C > 0$,

$$y_T(g) \leq C \cdot \mathrm{H}_{\iota,S}(g)^{-a_\iota} \quad \text{for any } g \in \mathbf{G}_S. \tag{4.29}$$

By (4.15), for any $h \in \mathbf{G}^S$, there exists a constant $c_h \geq 1$ such that for any $T > 2$,

$$c_h^{-1} \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \leq \tau^S(B_T \cap hG_{W_f} \cap \mathbf{G}^S) \leq c_h \cdot T^{a_\iota} (\log T)^{b_\iota - 1}. \tag{4.30}$$

Since $\{s_g \in \mathbf{G}^S : g \in \mathbf{G}_S\}$ is a finite set, $c := \max\{c_{s_g} : g \in \mathbf{G}_S\} < \infty$. It follows by (4.30) that for any $g \in \mathbf{G}_S$ with $\mathrm{H}_{\iota,S}(g) \leq T/2$,

$$\tau^S(B_{T \cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap s_g G_{W_f} \cap \mathbf{G}^S) \leq c \cdot \mathrm{H}_{\iota,S}(g)^{-a_\iota} T^{a_\iota} (\log T\, \mathrm{H}_{\iota,S}(g)^{-1})^{b_\iota - 1}.$$

On the other hand for any $g \in \mathbf{G}_S$ satisfying $T/2 \leq \mathrm{H}_{\iota,S}(g) \leq T\delta_0^{-1}$ where $\delta_0 := \inf_{g \in \mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g) > 0$ (see Lemma 2.5),

$$\tau^S(B_{T \cdot \mathrm{H}_{\iota,S}(g)^{-1}} \cap s_g G_{W_f} \cap \mathbf{G}^S) \leq \tau^S(B_2 \cap \mathbf{G}^S) \leq d \cdot \mathrm{H}_{\iota,S}(g)^{-a_\iota} T^{a_\iota}.$$

where $d = \delta_0^{-a_\iota} \tau^S(B_2 \cap \mathbf{G}^S)$. Also, for $\mathrm{H}_{\iota,S}(g) > T\delta_0^{-1}$, $y_T(g) = 0$.

Hence by applying (4.30) once more now to $h = e$, we obtain the inequality (4.29). Since $\mathrm{H}_{\iota,S}^{-a_\iota} \in L^1(\mathbf{G}_S)$ by Lemma 4.2, it follows that y_T belongs to $L^1(\mathbf{G}_S)$. Since by Theorem 4.13,

$$y_T(g) \to \gamma_{W_f,S}(s_g^{-1})\, \mathrm{H}_{\iota,S}(g)^{-a_\iota} \gamma_{W_f}^S(e)^{-1} \quad \text{as } T \to \infty,$$

we apply the dominated convergence theorem to (4.28) and deduce that

$$\lim_{T \to \infty} \frac{\int_{B_T \cap G_{W_f}} \psi \, d\tau_{W_f}}{\tau^S(B_T \cap G_{W_f} \cap \mathbf{G}^S)} = \gamma_{W_f}^S(e)^{-1} \int_{\mathbf{G}_S} \psi(g) \gamma_{W_f}^S(s_g^{-1})\, \mathrm{H}_{\iota,S}(g)^{-a_\iota} \, d\tau_S(g).$$

Using $\gamma_{W_f}^S(s_g^{-1}) = \gamma_{W_f}^S(g)$ and the definition of the measure $\mu_{\iota,W_f,S}$, we have

$$\lim_{T \to \infty} \frac{\int_{B_T \cap G_{W_f}} \psi \, d\tau_{W_f}}{\tau^S(B_T \cap G_{W_f} \cap \mathbf{G}^S)} = \tilde{\mu}_\iota(\psi) \cdot \gamma_{W_f}^S(e)^{-1} \tag{4.31}$$

Taking $\psi = 1$, we also get

$$\lim_{T \to \infty} \frac{\tau_{W_f}(B_T \cap G_{W_f})}{\tau^S(B_T \cap G_{W_f} \cap \mathbf{G}^S)} = \gamma_{W_f}^S(e)^{-1}. \tag{4.32}$$

Therefore combining (4.31) and (4.32), we obtain

$$\lim_{T \to \infty} \frac{\int_{B_T \cap G_{W_f}} \psi \, d\tau_{W_f}}{\tau_{W_f}(B_T \cap G_{W_f})} = \tilde{\mu}_\iota(\psi).$$

□

5. Equidistribution for saturated cases

Let $\mathbf{G}$ be a connected adjoint semisimple group defined over K and ι be a faithful representation of $\mathbf{G}$ to GL_N defined over K which has a unique maximal weight. We recall that $\iota : \mathbf{G} \to \mathrm{GL}_N$ is called *saturated* if the set

$$\{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\} \tag{5.1}$$

is not contained in the root system of a proper normal subgroup of $\mathbf{G}$.

Fix a height function H_ι on $\mathbf{G}(\mathbb{A})$ associated to ι, and let $\tilde{\mu}_\iota$ be the probability measure on X_ι constructed in Theorem 4.18 (associated to H_ι).

Theorem 5.2. *Suppose that ι is saturated. For $\psi \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{\tau_{W_f}(B_T \cap G_{W_f})} \sum_{g\in\mathbf{G}(K):\mathrm{H}_\iota(g)<T} \psi(g) = \int_{X_\iota} \psi \, d\tilde{\mu}_\iota$$

where W_f is any-cofinite subgroup of W_ι.

Note that by Corollary 4.16, the above equality is independent of W_f. To derive the asymptotic formula for the number of K-rational points, it suffices to take $\psi = 1$ in Theorem 5.2, and hence we obtain Theorem 1.10.

Corollary 5.3. *If ι is saturated, we have, as $T \to \infty$,*

$$\#\{g \in \mathbf{G}(K) : \mathbf{H}_\iota(g) < T\} \sim \tau_{W_f}(B_T \cap G_{W_f})$$

for any co-finite subgroup W_f of W_ι.

Combining Theorem 5.2 with Theorem 4.13, we deduce Theorems 1.2 (without an error term) and 1.6 for the saturated case. The rate of convergence as well as nonsaturated case are discussed in the next section.

Remark 5.4. Note that for any finite S, the projection of $\tilde{\mu}_\iota$ to $X_{\iota,S}$ is $\tilde{\mu}_{\iota,S}$, which is equivalent to a Haar measure on $\mathbf{G}_S$ by Proposition 4.22. Any open subset X_ι contains a subset of the form $V_S X_{R-S}$ where V_S is an open subset of $X_{\iota,S}$, which again contains $(gW_f \cap \mathbf{G}_S)\mathbf{G}^S$ for some $g \in \mathbf{G}_S$ and some co-finite subgroup W_f of W_ι. Now

$$\tilde{\mu}_\iota((gW_f \cap \mathbf{G}_S)\mathbf{G}^S) = \tilde{\mu}_{\iota,S}(gW_f \cap \mathbf{G}_S) > 0.$$

This shows that $\tilde{\mu}_\iota$ has full support on X_ι.

The rest of section is devoted to a proof of Theorem 5.2. We recall the following facts:

Lemma 5.5. *Let $\mathbf{G}$ be a connected semisimple adjoint K-group, $\iota : \mathbf{G} \to \mathrm{GL}(V)$ a K-rational representation of $\mathbf{G}$, and let $\mathbf{M}$ be a connected normal K-subgroup of $\mathbf{G}$.*

(1) *There exists a connected normal K-subgroup $\mathbf{N}$ of $\mathbf{G}$ so that $\mathbf{G} = \mathbf{MN}$ and $\mathbf{M} \cap \mathbf{N} = \{e\}$, and $\mathbf{M}$ is semisimple adjoint.*

(2) *For each* $x \in \mathbf{N}(\mathbb{A})$, *the function* $g \mapsto \mathrm{H}_\iota(gx)$ *defines a height function on* $\mathbf{M}(\mathbb{A})$ *with respect to the restriction* $\iota|_{\mathbf{M}}$.
(3) *If* ι *has a unique maximal weight* λ_ι, *then the restriction* $\iota|_{\mathbf{M}}$ *has a unique maximal weight.*

Proof. (1) follows directly from Proposition 3.28. Let $x \in \mathbf{N}(\mathbb{A})$. Whenever $x \in \mathbf{N}(K_v) \cap W_\iota$, which is the case for almost all v, we have $\mathbf{H}_{\iota,v}(gx) = \mathbf{H}_{\iota,v}(g)$ for all $g \in \mathbf{M}(K_v)$. Using this, it is easy to verify that the function $g \mapsto \mathrm{H}_\iota(gx)$ is a height function as defined in Definition 2.3.

For (3), let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ defined over K, Π denote the set of all weights of ι with respect to $\mathbf{T}$, and Π' denote the set obtained by restricting elements of Π to $\mathbf{M}\cap\mathbf{T}$. If $V = \oplus_{\lambda\in\Pi} V_\lambda$ is the weight space decomposition for ι, the weight space decomposition for $\iota|_{\mathbf{M}}$ is of the form $V = \oplus_{\beta\in\Pi'} W_\beta$ where $W_\beta := \oplus\{V_\lambda : \lambda|_{\mathbf{T}\cap\mathbf{M}} = \beta\}$. In particular, any weight of $\iota|_{\mathbf{M}}$ is the restriction of a weight of ι to $\mathbf{T}\cap\mathbf{M}$. Hence if β_ι is the restriction of λ_ι to $\mathbf{T}\cap\mathbf{M}$, and $\beta \in \Pi'$, not equal to β_ι then $\beta_\iota - \beta$ is a non-zero sum of positive roots of $\mathbf{M}$ with respect to $\mathbf{M}\cap\mathbf{T}$. Therefore β_ι is the unique maximal weight of $\iota|_{\mathbf{M}}$. □

Lemma 5.6. *The following are equivalent.*

(1) ι *is saturated.*
(2) *For any proper connected normal* K*-subgroup* $\mathbf{M}$ *of* $\mathbf{G}$,
$$\tau_{\mathbf{M}}(B_T \cap \mathbf{M}(\mathbb{A})) = O(T^{a_\iota}(\log T)^{b_\iota - 2})$$
where $\tau_{\mathbf{M}}$ *is a Haar measure on* $\mathbf{M}(\mathbb{A})$.

Proof. Assume (1). Since ι has a unique maximal weight, the restriction $\iota|_{\mathbf{M}}$ of ι has a unique maximal weight as well. There exists a connected normal K-subgroup $\mathbf{N}$ of $\mathbf{G}$ so that $\mathbf{G} = \mathbf{MN}$ and $\mathbf{M}\cap\mathbf{N} = \{e\}$. By Lemma 2.4, without loss of generality, we may assume that H_ι is the product of height functions $\mathrm{H}_{\iota|_{\mathbf{M}}}$ and $\mathrm{H}_{\iota|_{\mathbf{N}}}$ and $\mathrm{H}_\iota(e) = 1$. Hence $B_T \cap \mathbf{M}(\mathbb{A}) = \{x \in \mathbf{M}(\mathbb{A}) : \mathrm{H}_{\iota|_{\mathbf{M}}}(x) \le T\}$. Hence by Theorem 4.13,
$$\tau_{\mathbf{M}}(B_T \cap \mathbf{M}(\mathbb{A})) \sim c \cdot T^a (\log T)^{b-1}$$
where a and b are defined as in (4.1) for $\iota|_{\mathbf{M}}$. Now the saturated condition means that $a = a_\iota$ and $b \le b_\iota - 1$. Hence (2) holds. To show the other direction, suppose that ι is not saturated. Then there exists connected proper normal K-subgroups $\mathbf{M}$ and $\mathbf{N}$ of $\mathbf{G}$ as above such that $\{\alpha \in \Delta : u_\alpha + 1 = a_\iota \cdot m_\alpha\}$ is contained in the root system of $\mathbf{M}$. By Theorem 4.13, there exists $c > 0$ such that for all large T,
$$\tau_{\mathbf{M}}(B_T \cap \mathbf{M}(\mathbb{A})) \ge c \cdot T^{a_\iota}(\log T)^{b_\iota - 1}.$$
Hence (2) does not hold. □

The following lemma is the main reason why we need the assumption of ι being saturated for the proof of Theorem 5.2.

By Lemma 5.5, $\mathbf{G}$ is a product of connected K-simple adjoint subgroups.

Lemma 5.7. *Suppose that ι is saturated. Write $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_m$ as a product of connected K-simple subgroups. Then for any fixed $C > 0$,*

$$\lim_{T\to\infty} \frac{\tau_{W_f}((B_T - B^C) \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} = 0$$

where $B^C := \{(g_1, \cdots, g_m) \in \mathbf{G}(\mathbb{A}) : \mathrm{H}_\iota(g_i) > C \text{ for each } i = 1, \cdots, m\}$.

Proof. Since G_{W_f} is non-compact, $\tau_{W_f}(B_T \cap G_{W_f}) \to \infty$ as $T \to \infty$. If $m = 1$, the claim follows immediately from this. Suppose $m \geq 2$. Without loss of generality, we may assume that $\mathrm{H}_\iota(g_1, \cdots, g_m) = \prod_{i=1}^m \mathrm{H}_\iota(g_i)$. For each i, let B_i^C denote the subset of B_T consisting of $g = (g_1, \cdots, g_m)$ with $\mathrm{H}_\iota(g_i) \leq C$. If we denote by τ_i and τ^i Haar measures on $\mathbf{G}_i(\mathbb{A})$ and $\prod_{j\neq i} \mathbf{G}_j(\mathbb{A})$ such that $\tau_{W_f} = (\tau_i \times \tau^i)|_{G_{W_f}}$, then

$$\tau_{W_f}(B_i^C) \leq C_0 \cdot \tau^i(\{g' \in \prod_{j\neq i} \mathbf{G}_j(\mathbb{A}) : \mathrm{H}_\iota(g') < \delta_i^{-1} \cdot T\})$$

where $\delta_i = \inf_{g\in\mathbf{G}_i(\mathbb{A})} \mathrm{H}_\iota(g)$ and $C_0 = \tau_i(\{g \in \mathbf{G}_i(\mathbb{A}) : \mathrm{H}_\iota(g) \leq C\})$.

By the previous lemma, for some $c_i > 0$,

$$\tau^i(\{g' \in \prod_{j\neq i} \mathbf{G}_j(\mathbb{A}) : \mathrm{H}_\iota(g') < \delta_i^{-1} \cdot T\}) = O(T^{a_\iota} (\log T)^{b_\iota - 2}).$$

Hence for each i,

$$\lim_{T\to\infty} \frac{\tau_{W_f}(B_i^C \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} = 0.$$

Since $B_T - B^C \subset \cup_{i=1}^m B_i^C$, this proves the claim. $\square$

In the following, we fix a co-finite subgroup W_f of W_ι. Set

$$Y_{W_f} = \mathbf{G}(K)\backslash G_{W_f}.$$

For a fixed $\psi \in C(X_\iota)^{W_f}$, we define a function F_T on $G_{W_f} \times G_{W_f}$ by

$$F_T(g, h) = \sum_{\gamma\in\mathbf{G}(K)} \psi(g^{-1}\gamma h) \cdot \chi_{B_T}(g^{-1}\gamma h).$$

Since B_T is a compact subset of $\mathbf{G}(\mathbb{A})$, the above sum is finite and since F_T is $\mathbf{G}(K) \times \mathbf{G}(K)$-invariant, we may consider F_T as a function on $Y_{W_f} \times Y_{W_f}$

Note that

$$F_T(e, e) = \sum_{\gamma\in\mathbf{G}(K):\mathrm{H}_\iota(\gamma)\leq T} \psi(\gamma).$$

Proposition 5.8 (Weak-convergence)**.** *Suppose that ι is saturated. For $i = 1, 2$, let $\alpha_i \in C(Y_{W_f})$ be a W_f-invariant function and $\int_{Y_{W_f}} \alpha_i \, d\tau_{W_f} = 1$. If $\alpha(x, y) := \alpha_1(x)\alpha_2(y)$, then*

$$\lim_{T\to\infty} \frac{1}{\tau_{W_f}(B_T \cap G_{W_f})} \int_{Y_{W_f}\times Y_{W_f}} F_T \cdot \alpha \; d(\tau_{W_f} \times \tau_{W_f}) = \int_{X_\iota} \psi \, d\tilde{\mu}_\iota.$$

Proof. Observe that α is $W_f \times W_f$-invariant and

$$\begin{aligned}
(5.9)\quad & \langle F_T, \alpha\rangle_{Y_{W_f}\times Y_{W_f}} \\
&= \int_{x\in Y_{W_f}} \int_{y\in Y_{W_f}} \left(\sum_{\gamma\in\mathbf{G}(K)} \psi(x^{-1}\gamma y)\chi_{B_T}(x^{-1}\gamma y)\right) \alpha_1(x)\alpha_2(y)\, d\tau_{W_f}(y) d\tau_{W_f}(x) \\
&= \int_{x\in Y_{W_f}} \int_{h\in G_{W_f}} \psi(x^{-1}h)\chi_{B_T}(x^{-1}h)\alpha_1(x)\alpha_2(h)\, d\tau_{W_f}(h) d\tau_{W_f}(x) \\
&= \int_{g\in G_{W_f}} \psi(g)\chi_{B_T}(g) \left(\int_{x\in Y_{W_f}} \alpha_1(x)\alpha_2(xg)\, d\tau_{W_f}(x)\right) d\tau_{W_f}(g) \\
&= \int_{g\in B_T\cap G_{W_f}} \psi(g)\langle \alpha_1, g\cdot\alpha_2\rangle\, d\tau_{W_f}(g).
\end{aligned}$$

As in the above lemma 5.7, we write $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_m$. Now for a sequence $\{g \in \mathbf{G}(\mathbb{A})\}$, that $g \to \infty$ strongly means precisely the $\mathbf{G}_i(\mathbb{A})$-component g_i of g tends to ∞ for each $1 \le i \le m$. Since the height function H_ι is proper, this is again equivalent to saying that $\mathrm{H}_\iota(g_i) \to \infty$ for each $i = 1, \dots, m$.

For $i = 1, 2$, we claim that $\alpha_i - 1 \in L^2_{00}(Y_{W_f})$, that is, for any automorphic character χ of $\mathbf{G}(\mathbb{A})$,

$$\int_{Y_{W_f}} \alpha\cdot\chi\, d\tau_{W_f} = \int_{Y_{W_f}} \chi\, d\tau_{W_f}.$$

If χ is W_f-invariant, then $\chi = 1$ on G_{W_f}. Since $\int_{Y_{W_f}} \alpha\, d\tau_{W_f} = 1$, the claim is clear. If $\chi \in \Lambda - \Lambda^{W_f}$, we only need to apply Lemma 4.12 for $\psi = \alpha - 1_{Y_f}$, where 1_{Y_f} is the characteristic function of Y_{W_f}, since we may consider ψ as a function on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ which is 0 outside Y_{W_f}.

Hence, by Theorem 3.26, for any given $\epsilon > 0$, there exists $C > 0$ such that for all $g \in B^C$,

$$|\langle \alpha_1, g\cdot\alpha_2\rangle - 1| = |\langle \alpha_1 - 1, g\cdot(\alpha_2 - 1)\rangle| < \epsilon \tag{5.10}$$

Hence

$$\left|\int_{g\in B_T\cap G_{W_f}} \psi(g)\langle\alpha_1, g.\alpha_2\rangle\, d\tau_{W_f}(g) - \int_{g\in B_T\cap G_{W_f}} \psi(g)\, d\tau_{W_f}(g)\right|$$
$$< \sup|\psi|\cdot(\|\alpha_1\|\cdot\|\alpha_2\| + 1)\cdot\tau_{W_f}((B_T - B^C)\cap G_{W_f}) + \epsilon\cdot\sup|\psi|\cdot\tau_{W_f}(B_T\cap B^C\cap G_{W_f})$$

where $\|\alpha_i\|$ is the L^2-norm of $\alpha_i \in L^2(Y_{W_f})$ for each i. By Lemma 5.7, it follows that

$$\limsup_{T\to\infty} \frac{1}{\tau_{W_f}(B_T\cap G_{W_f})}\left|\int_{g\in B_T\cap G_{W_f}} \psi(g)(\langle\alpha_1, g.\alpha_2\rangle - 1)\, d\tau_{W_f}(g)\right| \le \epsilon\cdot\sup|\psi|.$$

Since $\epsilon > 0$ is arbitrary, by (5.9), this proves

$$\lim_{T\to\infty} \frac{1}{\tau_{W_f}(B_T \cap G_{W_f})} \int_{Y_{W_f}\times Y_{W_f}} F_T \cdot \alpha \; d(\tau_{W_f} \times \tau_{W_f}) = \lim_{T\to\infty} \frac{\int_{B_T\cap G_{W_f}} \psi \, d\tau_{W_f}}{\tau_{W_f}(B_T \cap G_{W_f})}.$$

By Lemma 4.27, this proves Proposition 5.8. □

Proof of Theorem 5.2. It suffices to prove our theorem for nonnegative functions $\psi \in C(X_\iota)^{W_f}$ for each co-finite subgroup W_f of W_ι. Fix $\epsilon > 0$. Let W_∞ be a symmetric neighborhood of e in $\mathbf{G}_\infty^\circ$ such that

$$W_\infty B_T W_\infty \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \bigcap_{g,h\in W_\infty} gB_T h \quad \text{for all } T > 1.$$

By the uniform continuity of ψ, replacing W_∞ by a smaller one if necessary, we may assume that

$$\psi(g^{-1}xh) - \epsilon \le \psi(x) \le \psi(g^{-1}xh) + \epsilon \quad \text{for all } x \in X \text{ and } g, h \in W_\infty. \tag{5.11}$$

Define

$$F_T^\pm(g,h) = \sum_{\gamma\in\mathbf{G}(K)} (\psi(g^{-1}\gamma h) \pm \epsilon) \cdot \chi_{B_T}(g^{-1}\gamma h).$$

We claim that for any $T > 1$ and for any $g, h \in W := W_\infty \times W_f$,

$$F^-_{(1-\epsilon)T}(g,h) \le F_T(e,e) \le F^+_{(1+\epsilon)T}(g,h). \tag{5.12}$$

To see this, observe that if $\gamma \in \mathbf{G}(K)$ with $\mathrm{H}_\iota(\gamma) < T$, and $g, h \in W$ then

$$\mathrm{H}_\iota(g^{-1}\gamma h) \le (1+\epsilon)T \quad \text{and} \quad f(\gamma) \le \psi(g^{-1}\gamma h) + \epsilon.$$

Hence

$$F_T(e,e) = \sum_{\gamma\in\mathbf{G}(K),\mathrm{H}_\iota(\gamma)<T} \psi(\gamma) \le \sum_{\gamma\in\mathbf{G}(K),\mathrm{H}_\iota(g^{-1}\gamma h)<(1+\epsilon)T} (\psi(g^{-1}\gamma h)+\epsilon) = F^+_{(1+\epsilon)T}(g,h),$$

proving the right inequality in (5.12). The other inequality can be proved similarly. Now let $\phi \in C_c(Y_{W_f})$ be a non-negative W_f-invariant function such that $\mathrm{supp}(\phi) \subset \mathbf{G}(K)\backslash\mathbf{G}(K)W$ and $\int_{Y_{W_f}} \phi \, d\tau_{W_f} = 1$. By integrating (5.12) over $Y_{W_f} \times Y_{W_f}$ against the function $\alpha(x,y) = \phi(x)\cdot\phi(y)$, we obtain

$$\langle F^-_{(1-\epsilon)T}, \alpha\rangle \le F_T(e,e) \le \langle F^+_{(1+\epsilon)T}, \alpha\rangle.$$

Note that Theorem 4.13 implies the following: there exist sequences $\{a_\epsilon \ge 1\}$ and $\{b_\epsilon \le 1\}$ such that $a_\epsilon \to 1$ and $b_\epsilon \to 1$ as $\epsilon \to 0$ for all sufficiently small $\epsilon > 0$,

$$b_\epsilon \le \liminf_T \frac{\tau_{W_f}(B_{(1-\epsilon)T} \cap G_{W_f}))}{\tau_{W_f}(B_T \cap G_{W_f})} \le \limsup_T \frac{\tau_{W_f}(B_{(1+\epsilon)T} \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} \le a_\epsilon. \tag{5.13}$$

Hence by applying Proposition 5.8,

$$\begin{aligned}
\limsup_T \frac{F_T(e,e)}{\tau_{W_f}(B_T \cap G_{W_f}))} &\le \limsup_T \frac{\langle F^+_{(1+\epsilon)T}, \alpha\rangle}{\tau_{W_f}(B_T \cap G_{W_f})} \\
&\le \limsup_T \frac{\langle F^+_{(1+\epsilon)T}, \alpha\rangle}{\tau_{W_f}(B_{(1+\epsilon)T} \cap G_{W_f})} \cdot \limsup_T \frac{\tau_{W_f}(B_{(1+\epsilon)T} \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} \\
&\le a_\epsilon \cdot \int_{X_\iota} (\psi + \epsilon) d\mu_{\iota, W_f} \le a_\epsilon \cdot (\int_{X_\iota} \psi d\mu_{\iota, W_f} + \epsilon)
\end{aligned}$$

and similarly,

$$b_\epsilon \cdot (\int_{X_\iota} \psi d\tilde{\mu}_\iota - \epsilon) \le \liminf_T \frac{F_T(e,e)}{\tau_{W_f}(B_T \cap G_{W_f})}.$$

Taking $\epsilon \to 0$,

$$\lim_T \frac{F_T(e,e)}{\tau_{W_f}(B_T \cap G_{W_f})} = \int_{X_\iota} \psi \, d\mu_{\iota, W_f}.$$

This finishes the proof of Theorem 5.2. □

In fact, for the case $\psi = 1$, the computation in the proof of Theorem 5.2 can be simplified significantly, and it applies to general families of balls B_T, which we presently introduce.

For an increasing sequence $\{B_T\}$ of relatively compact subsets of $\mathbf{G}(\mathbb{A})$ and a compact open subgroup $W_f \subset \mathbf{G}(\mathbb{A}_f)$, we call $\{B_T\}$ W_f-*well rounded* if the following holds:

(1) $W_f B_T W_f = B_T$ for any $T > 1$;
(2) for any small $\epsilon > 0$, there exists a neighborhood $W_\epsilon \subset \mathbf{G}^\circ_\infty$ of e such that

$$W_\epsilon B_T W_\epsilon \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \bigcap_{g,h \in W_\epsilon} g B_T h$$

for all $T > 1$;
(3) $\tau_{W_f}(B_T \cap G_{W_f}) \to \infty$ as $T \to \infty$ and there exist constants $a_\epsilon \ge 1$ and $b_\epsilon \le 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$,

$$b_\epsilon \le \liminf_T \frac{\tau_{W_f}(B_{(1-\epsilon)T} \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} \le \limsup_T \frac{\tau_{W_f}(B_{(1+\epsilon)T} \cap G_{W_f})}{\tau_{W_f}(B_T \cap G_{W_f})} \le a_\epsilon.$$

The proof of Theorem 5.2 gives

Proposition 5.14. *Let $\mathbf{G}$ be a connected absolutely almost simple K-group, and let W_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then for any W_f-well rounded sequence $\{B_T\}$ of relatively compact subsets of $\mathbf{G}(\mathbb{A})$,*

$$\#\mathbf{G}(K) \cap B_T \sim_{T\to\infty} \tau_{W_f}(B_T \cap G_{W_f}).$$

6. Arithmetic fibrations and Construction of μ_ι

In this section we prove the main theorems for a general case, that is, without the saturation assumption on ι. We let $\mathbf{G}$ be a connected semisimple adjoint group defined over K and ι a faithful representation of $\mathbf{G}$ to GL_N defined over K which has a unique maximal weight λ_ι. Fix a height function H_ι on $\mathbf{G}(\mathbb{A})$ associated to ι.

Let $\mathbf{T}$, $\Phi(\mathbf{G},\mathbf{T})$, Δ, $2\rho = \sum_{\alpha\in\Delta} u_\alpha \alpha$, $\lambda_\iota = \sum_{\alpha\in\Delta} m_\alpha \alpha$ and a_ι, b_ι be as defined in Section 1.2. Let $\mathbf{M}$ be the smallest connected normal K-subgroup of $\mathbf{G}$ whose root system contains the set

$$\{\alpha\in\Delta : \frac{u_\alpha+1}{m_\alpha} = a_\iota\}.$$

Let $\mathbf{N}$ be a connected normal K-subgroup of $\mathbf{G}$ such that $\mathbf{G}=\mathbf{M}\mathbf{N}$ and $\mathbf{M}\cap\mathbf{N}=\{e\}$. Let $\pi:\mathbf{G}\to\mathbf{N}$ be the canonical projection. Note that any element of $\mathbf{G}(\mathbb{A})$ can be uniquely written as g_1g_2 with $g_1\in\mathbf{M}(\mathbb{A})$ and $g_2\in\mathbf{N}(\mathbb{A})$. We denote by $\tau_{\mathbf{M}}$ the Haar measure on $\mathbf{M}(\mathbb{A})$ such that $\tau_{\mathbf{M}}(\mathbf{M}(\mathbb{A})/\mathbf{M}(K))=1$.

For each $x\in\mathbf{N}(\mathbb{A})$, the function $\mathrm{H}_\iota^x(g) := \mathrm{H}_\iota(gx)$ defines a height function on $\mathbf{M}(\mathbb{A})$ with respect to $\iota' := \iota|_{\mathbf{M}}$, and ι' has a unique maximal weight by Lemma 5.5. Also the definition of $\mathbf{M}$ implies that ι' is saturated, $a_{\iota'}=a_\iota$ and $b_{\iota'}=b_\iota$. Set

$$V_\iota := \mathbf{M}(\mathbb{A}_f)\cap W_\iota.$$

Then Theorems 1.2 and 5.2 (for the saturated cases) imply that for each $x\in\mathbf{N}(K)$,

(1) there exists $c_x>0$ such that

$$N_{\pi^{-1}(x)}(\mathrm{H}_\iota,T) = \#\{g\in\mathbf{M}(K) : \mathrm{H}_\iota(gx)<T\} \sim c_x\cdot T^{a_\iota}(\log T)^{b_\iota-1}; \tag{6.1}$$

(2) the following number

$$r_{x,\iota} = \sum_{\chi\in\Lambda}\lim_{s\to a_\iota^+}(s-a_\iota)^{b_\iota}\int_{\mathbf{M}(\mathbb{A})}\mathrm{H}_\iota(gx)^{-s}\chi(g)\,d\tau_{\mathbf{M}}(g)$$

is a positive real number;

(3) if $X_{\mathbf{M}}$ denotes the closed subspace $\prod_{v\in R}\overline{\iota'(\mathbf{M}(K_v))}$ of X_ι, there exists a probability measure $\tilde\mu_{x,\iota}$ on $X_{\mathbf{M}}$ such that for any $\psi\in C(X_{\mathbf{M}})$ which is invariant under a co-finite subgroup of V_ι,

$$\tilde\mu_{x,\iota}(\psi) = r_{x,\iota}^{-1}\cdot\sum_{\chi\in\Lambda}\lim_{s\to a_\iota^+}(s-a_\iota)^{b_\iota}\int_{\mathbf{M}(\mathbb{A})}\mathrm{H}_\iota(gx)^{-s}\chi(g)\,\psi(g)d\tau_{\mathbf{M}}(g).$$

Noting that $N(\mathrm{H}_\iota,T) = \sum_{x\in\mathbf{N}(K)} N_{\pi^{-1}(x)}(\mathrm{H}_\iota,T)$, we restate Theorem 1.2 in the introduction.

Theorem 6.2. *We have*

(1) $c_{\mathrm{H}_\iota} := \sum_{x\in\mathbf{N}(K)} c_x < \infty$;

(2) *for some $\delta > 0$,*

$$N(\mathrm{H}_\iota, T) = c_{\mathrm{H}_\iota} \cdot T^{a_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta})). \tag{6.3}$$

Set for $T > 0$ and $x \in \mathbf{N}(\mathbb{A})$,

$$B_T^x = \{g \in \mathbf{M}(\mathbb{A}) : \mathrm{H}_\iota^x(g) < T\}.$$

Since x commutes with $\mathbf{M}(\mathbb{A})$, each height function H_ι^x on $\mathbf{M}(\mathbb{A})$ is invariant under V_ι. Let $Y_{\mathbf{M}} = \mathbf{M}(K) \backslash M_{V_\iota}$ and τ be the invariant probability measure on $Y_{\mathbf{M}}$. For each $x \in \mathbf{N}(K)$, set

$$F_T^x(g, h) := \sum_{\gamma \in \mathbf{M}(K)} \chi_{B_T^x}(g^{-1} \gamma h), \quad g, h \in \mathbf{M}(\mathbb{A}).$$

We may consider F_T^x as a function on $Y_{\mathbf{M}} \times Y_{\mathbf{M}}$. Write $\mathbf{M} = \mathbf{M}_1 \cdots \mathbf{M}_r$ as a product of connected K-simple K-groups and denote by τ_i the invariant probability measure on $\mathbf{M}_i(K) \backslash \mathbf{M}_i(\mathbb{A}) \cap M_{V_\iota}$. For a collection of smooth $(W_f \cap \mathbf{M}_i(\mathbb{A}))$-invariant functions $\psi_i \in C_c(\mathbf{M}_i(K) \backslash \mathbf{M}_i(\mathbb{A}) \cap M_{V_\iota})$, such that $\int \psi_i \, d\tau_i = 1$ for each $1 \le i \le r$, define $\psi \in C_c(Y_{\mathbf{M}})$ and $\alpha \in C_c(Y_{\mathbf{M}} \times Y_{\mathbf{M}})$ by

$$\psi(z_1, \cdots, z_r) := \prod_{i=1}^r \psi_i(z_i) \quad \text{and} \quad \alpha(y_1, y_2) := \psi(y_1) \psi(y_2).$$

Lemma 6.4. (1) *There is a constant $c > 0$ such that for any $x \in \mathbf{N}(K)$,*

$$c_x \le c \cdot \mathrm{H}_\iota(x)^{-a_\iota}.$$

(2) *There exist $l \in \mathbb{N}$ and $\delta > 0$, independent of x, such that for any $x \in \mathbf{N}(K)$ and $T \gg \mathrm{H}_\iota(x)$,*

$$\langle F_T^x, \alpha \rangle_{Y_{\mathbf{M}} \times Y_{\mathbf{M}}} = c_x \cdot T^{a_\iota} (\log T)^{b_\iota - 1} + O(d_x \cdot C'_\psi \cdot T^{a_\iota} (\log T)^{b_\iota - 1 - \delta})$$

where $d_x = \mathrm{H}_\iota(x)^{-a_\iota} (\log \mathrm{H}_\iota(x))^{b_\iota - 1}$ and $C'_\psi = \max(1, \max_i \|\mathcal{D}^l \psi_i\|^{2r})$ and $\mathcal{D}$ is the elliptic operator defined in (3.24).

Proof. As in the proof of Proposition 5.8, we derive that

$$\langle F_T^x, \alpha \rangle = \int_{g \in B_T^x \cap M_{V_\iota}} \langle \psi, g.\psi \rangle \, d\tau(g)$$

Note that

$$\begin{aligned}
|\langle\psi, g.\psi\rangle - 1| &= |\prod_{i=1}^{r}\langle\psi_i, g_i.\psi_i\rangle - 1| \\
&= |\sum_{i=1}^{r}(\prod_{j=1}^{i-1}\langle\psi_j, g_j.\psi_j\rangle)(\langle\psi_i, g_i.\psi_i\rangle - 1)| \\
&\le r\cdot C_\psi \cdot \max_i |\langle\psi_i, g_i.\psi_i\rangle - 1| \\
&= r\cdot C_\psi \cdot \max_i |\langle\psi_i - 1, g_i.(\psi_i - 1)\rangle|
\end{aligned}$$

where $C_\psi = \max(1, \max_i \|\psi_i\|^{2r-2})$. Since $\psi_i - 1 \in L^2_{00}(\mathbf{M}_i(K)\backslash\mathbf{M}_i(\mathbb{A})\cap M_{V_\iota})$ for each i, we deduce from Theorem 3.25 that

(6.5)

$$|\langle F_T^x, \alpha\rangle - \tau(B_T^x \cap M_{V_\iota})| \le 2r\cdot(\prod_i c_{W_f\cap\mathbf{M}_i(\mathbb{A})})\cdot C'_\psi \cdot \int_{g=g_1\cdots g_r\in B_T^x\cap M_{V_\iota}} (\max_i\ \tilde{\xi}_{\mathbf{M}_i}(g_i)^{1/2})\, d\tau(g)$$

where $C'_\psi = \max(1, \max_i \|\mathcal{D}^l\psi_i\|^{2r})$ for some large l.

Since $\tilde{\xi}_{\mathbf{M}_i} \le {\xi_{\mathbf{M}_i}}^{1/2}$, it follows from Lemma 3.6 that there exist $m\in\mathbb{N}$ and $C_1 > 0$ such that for any $1\le i\le r$,

$$\tilde{\xi}^{1/2}_{\mathbf{M}_i}(g_i) < C_1\cdot \mathrm{H}_\iota(g_i)^{-1/m} \quad \text{for any } g_i\in\mathbf{M}_i(\mathbb{A}).$$

Define a function on $\mathrm{M}(\mathbb{A})$ by

$$\tilde{\mathrm{H}}(g_1\cdots g_r) := \min_i \mathrm{H}_\iota(g_i), \quad g_i\in\mathbf{M}_i(\mathbb{A}).$$

Let κ be as in Lemma 2.4 for $\mathbf{G}_1 = \mathbf{M}$ and $\mathbf{G}_2 = \mathbf{N}$ so that $B_T^x \subset B_{\kappa T\cdot \mathrm{H}_\iota(x)^{-1}}$. It then follows from (6.5) that

$$|\langle F_T^x, \alpha\rangle - \tau(B_T^x\cap M_{V_\iota})| < C_2\cdot C'_\psi\cdot \int_{B_{\kappa T\cdot \mathrm{H}_\iota(x)^{-1}}\cap M_{V_\iota}} \tilde{\mathrm{H}}(g)^{-1/m}\, d\tau(g) \tag{6.6}$$

for a constant $C_2 > 0$ independent of x.

Since ι' is saturated, by Lemma 5.6, for every proper normal K-subgroup $\mathbf{L}$ of $\mathbf{M}$,

$$\tau_{\mathbf{L}}(B_T\cap\mathbf{L}(\mathbb{A})) \ll (\log T)^{-1}\tau(B_T\cap M_{V_\iota})$$

where $\tau_{\mathbf{L}}$ is a Haar measure on $\mathbf{L}(\mathbb{A})$.

For each $C>1$, set

$$B^C = \{g\in\mathbf{M}(\mathbb{A}) : \tilde{\mathrm{H}}(g) > C\}.$$

Note that

$$(B_T - B^C)\cap M_{V_\iota} \subset \cup_{i=1}^r \Omega_i$$

where $\Omega_i = \{g = g_1\cdots g_r \in M_{V_\iota} : \mathrm{H}_\iota(g_i)\le C,\ \ \mathrm{H}_\iota(g) < T\}$. Now denoting by $\mathbf{L}^{(i)}$ the subgroup of $\mathbf{M}$ generated by $\mathbf{M}_1,\cdots,\mathbf{M}_{i-1},\mathbf{M}_{i+1},\cdots,\mathbf{M}_r$, let $\kappa_i > 1$ be a constant

as in Lemma 2.4 for $\mathbf{G}_1 = \mathbf{M}_i$ and $\mathbf{G}_2 = \mathbf{L}^{(i)}$. Let $\delta_0 := \inf_{g\in\mathbf{G}(\mathbb{A})} \mathrm{H}_\iota(g) > 0$ (Lemma 2.5). Then for any $C \gg 1$,

$$\begin{aligned}\tau(\Omega_i) &\le \int_{\mathrm{H}_\iota(g_i)<C} \tau_{\mathbf{L}_i(\mathbb{A})}(B_{\kappa_i\delta_0^{-1}T} \cap \mathbf{L}^{(i)}_{V_\iota\cap\mathbf{L}(\mathbb{A}_f)})\, d\tau_{\mathbf{M}_i}(g_i)\\ &\ll C^{a_\iota}(\log C)^{b_\iota-1}(\log T)^{-1}\tau(B_{\kappa_0 T}\cap M_{V_\iota})\end{aligned}$$

where $\kappa_0 = \max_i(\kappa_i\delta_0^{-1})$.

Hence for any $C \gg 1$ and $T \gg C$,

$$\tau((B_T - B^C)\cap M_{V_\iota}) \ll C^{a_\iota}(\log C)^{b_\iota-1}(\log T)^{-1}\tau(B_{\kappa_0 T}\cap M_{V_\iota}).$$

Therefore

(6.7)

$$\begin{aligned}\int_{B_T\cap M_{V_\iota}} \tilde{\mathrm{H}}^{-1/m}\, d\tau &= \int_{B_T\cap B^C\cap M_{V_\iota}} \tilde{\mathrm{H}}^{-1/m}\, d\tau + \int_{(B_T-B^C)\cap M_{V_\iota}} \tilde{\mathrm{H}}^{-1/m}\, d\tau\\ &\ll (C^{-1/m} + \delta_0^{-1/m}\cdot C^{a_\iota}(\log C)^{b_\iota-1}(\log T)^{-1})\cdot\tau(B_{\kappa_0 T}\cap M_{V_\iota})\\ &\ll (\log T)^{-\delta}\cdot\tau(B_{\kappa_0 T}\cap M_{V_\iota}) \quad \text{for } C = (\log T)^{1/(2a_\iota)}\end{aligned}$$

for some $\delta > 0$. We now deduce from (6.6) and (6.7) that

$$\langle F_T^x, \alpha\rangle = \tau(B_T^x\cap M_{V_\iota}) + O\left(C'_\psi\cdot(\log T)^{-\delta}\cdot\tau(B_{\kappa_0\kappa T\cdot\mathrm{H}_\iota(x)^{-1}}\cap M_{V_\iota})\right) \tag{6.8}$$

for some $\delta > 0$. Let $S \subset R$ be as in the proof of Lemma 2.4, that is, for any $v \in R - S$,

$$\mathbf{G}(K_v) = U_v A_v^+ U_v \quad \text{and} \quad \mathrm{H}_v(\iota(g)) = \chi(a) \quad \text{for } g = u_1 a u_2 \in \mathbf{G}(K_v).$$

Denote by τ_S and τ^S Haar measures on $\mathbf{M}_S$ and $\mathbf{M}^S$ respectively such that $\tau = \tau_S\times\tau^S$ locally.

Recall from (4.9) the map $\mathbf{M}_S \to (M_{V_\iota}\cap\mathbf{M}^S)\backslash\mathbf{M}^S$ by $g\mapsto[s_g]$ and as in (4.28) we have

$$\begin{aligned}\tau(B_T^x\cap M_{V_\iota}) &= \int_{g\in\mathbf{M}_S} \tau^S(B_{\kappa T\cdot\mathrm{H}_\iota^{-1}(gx)}\cap s_g M_{V_\iota}\cap\mathbf{M}^S)\, d\tau_S(g).\\ &= \int_{g\in B_{\delta_0^{-1}\kappa T\cdot\mathrm{H}_\iota^{-1}(x)}\cap\mathbf{M}_S} \tau^S(B_{\kappa T\cdot\mathrm{H}_\iota^{-1}(gx)}\cap s_g M_{V_\iota}\cap\mathbf{M}^S)\, d\tau_S(g).\end{aligned} \tag{6.9}$$

By Theorem 4.13, there is $c_0 > 0$ such that for all $g\in\mathbf{M}_S$,

$$\tau^S(B_T\cap s_g M_{V_\iota}\cap\mathbf{M}^S) = c_0\cdot\gamma_{V_\iota,S}(s_g^{-1})\cdot T^{a_\iota}(\log T)^{b_\iota-1} + O(T^{a_\iota}(\log T)^{b_\iota-2}).$$

Here the implied constant can be taken uniformly for all $g\in\mathbf{G}_S$, since there are only finitely many cosets $s_g M_{V_\iota}\cap\mathbf{M}^S$.

Note that $\gamma^S_{V_\iota}(s_g^{-1}) = \gamma^S_{V_\iota}(g) = \gamma^S_{V_\iota}(gx)$ since $x \in \mathbf{N}(K)$ and it is bounded. We deduce that when $\mathrm{H}_\iota(gx) \ll T/\delta_0$,

$$\begin{aligned}&\tau^S(B_{\kappa T\,\mathrm{H}_\iota^{-1}(gx)} \cap M_{V_\iota} \cap \mathbf{M}^S)\\ =&c\cdot \gamma^S_{V_\iota}(g)(T\cdot \mathrm{H}_\iota^{-1}(gx))^{a_\iota}(\log T)^{b_\iota-1} + O((T\cdot \mathrm{H}_\iota^{-1}(gx))^{a_\iota}(\log \mathrm{H}_\iota(gx))^{b_\iota-1}(\log T)^{b_\iota-2}).\end{aligned}$$

for $c = c(S, V_\iota, \kappa) > 0$. To estimate the integral over the domain $\mathrm{H}_\iota(gx) \gg T/\delta_0$, it suffices to note that by Lemma 4.2,

$$\tau_S(B_{T\cdot \mathrm{H}_\iota^{-1}(x)} \cap \mathbf{M}_S) \ll (T\,\mathrm{H}_\iota^{-1}(x))^{a_\iota-\epsilon}.$$

Since by Lemmas 2.4 and 4.2,

$$\int_{g\in \mathbf{M}_S} \gamma^S_{V_\iota}(g)\,\mathrm{H}_\iota(gx)^{-a_\iota}(\log \mathrm{H}_\iota(gx))^{b_\iota-1}\,d\tau_S(g) \ll \mathrm{H}_\iota(x)^{-a_\iota}(\log \mathrm{H}_\iota(x))^{b_\iota-1},$$

it follows from the above estimates that for $T \gg \mathrm{H}_\iota(x)$,

$$\tau(B^x_T \cap M_{V_\iota}) = c_x T^{a_\iota}(\log T)^{b_\iota-1} + O(d_x T^{a_\iota}(\log T)^{b_\iota-2}),$$

where

$$c_x = c\cdot \int_{g\in \mathbf{M}_S} \gamma^S_{V_\iota}(gx)\,\mathrm{H}_\iota(gx)^{-a_\iota}\,d\tau_S(g) \ll \mathrm{H}_\iota(x)^{-a_\iota}. \tag{6.10}$$

Hence combining (6.8) and (6.9), we have for $T \gg \mathrm{H}_\iota(x)$,

$$\langle F^x_T, \alpha\rangle = c_x\cdot T^{a_\iota}(\log T)^{b_\iota-1} + O(d_x\cdot C'_\psi\cdot T^{a_\iota}(\log T)^{b_\iota-1-\delta}).$$

□

A key ingredient in deducing Theorem 6.2 is the following stronger version of (6.1):

Proposition 6.11. *There exists $\delta > 0$ such that for each $x \in \mathbf{N}(K)$ and for any $T \gg \mathrm{H}_\iota(x)$,*

$$N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T) = c_x\cdot T^{a_\iota}(\log T)^{b_\iota-1} + O(d_x\cdot T^{a_\iota}(\log T)^{b_\iota-1-\delta}) \tag{6.12}$$

where $d_x = \mathrm{H}_\iota(x)^{-a_\iota}(\log \mathrm{H}_\iota(x))^{b_\iota-1}$ and the implied constant is independent of x.

Proof. Let ϕ_ϵ be a smooth symmetric nonnegative function on $\mathbf{M}_\infty$, which is a product $\prod_{i=1}^r \phi_{i,\epsilon}$ of smooth functions on the simple factors of $\mathbf{M}_\infty$, $\int_{\mathbf{M}_\infty}\phi_\epsilon\,d\tau_\infty = 1$ and $\mathrm{supp}(\phi_\epsilon)$ is contained in the Riemannian ball at e in $\mathbf{M}_\infty$ of radius ϵ, and for some $\rho > 0$, $\max_i \|\mathcal{D}^l\phi_{i,\epsilon}\|^{2r} \ll \epsilon^{-\rho}$ (see, for example, Lemma 4.4 in [28]). By the definition of H_ι in (2.6), there exists $b > 0$ such that

$$\mathrm{supp}(\phi_\epsilon)\cdot B^x_T\cdot \mathrm{supp}(\phi_\epsilon) \subset B^x_{(1+b\epsilon)T}$$

for every $T > 1$ and $x \in \mathbf{N}(K)$.

Define

$$\psi_\epsilon(g) = \frac{1}{\tau^{R_f}(V_\iota)} \sum_{\gamma\in\mathbf{M}(K)} \phi_\epsilon(\gamma g_\infty)\cdot \chi_{V_\iota}(\gamma g_f), \quad g = g_\infty g_f \in \mathbf{M}_\infty\mathbf{M}(\mathbb{A}_f).$$

Define $\alpha_\epsilon(y_1, y_2) = \psi_\epsilon(y_1)\psi_\epsilon(y_2)$ for $(y_1, y_2) \in Y_{\mathbf{M}} \times Y_{\mathbf{M}}$. Then

$$\begin{aligned} &N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T) \le \left\langle F^x_{(1+b\epsilon)T}, \alpha_\epsilon \right\rangle \\ &= c_x T^{a_\iota} (\log T)^{b_\iota - 1} + O(c_x \cdot \epsilon \cdot T^{a_\iota} (\log T)^{b_\iota - 1} + d_x \cdot \epsilon^{-\rho} T^{a_\iota} (\log T)^{b_\iota - 1 - \delta}). \end{aligned}$$

Setting $\epsilon = (\log T)^{-\delta/(\rho+1)}$, we derive the upper estimate for $N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)$. The lower estimate is proved similarly. □

Proof of Theorem 6.2. According to the choice of $\mathbf{N}$, for any simple root $\alpha \in \Delta$ whose restriction to $\mathbf{N}$ is a root, we have

$$\frac{u_\alpha + 1}{m_\alpha} < a_\iota. \tag{6.13}$$

Since $\mathbf{N}(K)$ is a discrete subgroup of $\mathbf{N}(\mathbb{A})$, we can find an open neighborhood $U := U_\infty \times U_f$ of the identity in $\mathbf{N}(\mathbb{A})$ such that $\gamma U \cap \gamma' U = \emptyset$ for all $\gamma \neq \gamma' \in \mathbf{N}(K)$. We may assume $U_f \subset W_\iota$ and $B_T U_\infty \subset B_{2T}$ for all $T \gg 1$. Since $\tau_{\mathbf{N}}(\gamma U) = \tau_{\mathbf{N}}(U)$ by the invariance of $\tau_{\mathbf{N}}$, we deduce

$$N_{\mathbf{N}}(\mathrm{H}_\iota, T) = \tau_{\mathbf{N}}(U)^{-1} \cdot \tau_{\mathbf{N}} \left(\bigcup_{\gamma \in \mathbf{N}(K) : \mathrm{H}_\iota(x) \le T} \gamma U \right) \le \tau_{\mathbf{N}}(U)^{-1} \cdot \tau_{\mathbf{N}}(B_{2T} \cap \mathbf{N}(\mathbb{A})).$$

Therefore Theorem 4.13, applied to $\tau_{\mathbf{N}}(B_{2T} \cap \mathbf{N}(\mathbb{A}))$, together with (6.13) yields that there exists $\epsilon > 0$ such that

$$N_{\mathbf{N}}(\mathrm{H}_\iota, T) = O(T^{a_\iota - \epsilon}).$$

Hence setting $\alpha(t) = N_{\mathbf{N}}(\mathrm{H}_\iota, t)$, we have for any $a_\iota - \epsilon/2 \le a \le a_\iota$,

$$\sum_{x \in \mathbf{N}(K)} \mathrm{H}_\iota(x)^{-a} = \int_0^\infty t^{-a}\, d\alpha(t) = a \int_0^\infty t^{-a-1} \alpha(t)\, dt \ll \int_0^\infty t^{-(1+\epsilon/2)}\, dt < \infty.$$

Since $c_x \ll \mathrm{H}_\iota(x)^{-a_\iota}$ by (6.10) and $d_x = \mathrm{H}_\iota(x)^{-a_\iota} (\log \mathrm{H}_\iota(x))^{b_\iota - 1}$, it follows that

$$c_{\mathrm{H}_\iota} := \sum_{x \in \mathbf{N}(K)} c_x < \infty \quad \text{and} \quad \sum_{x \in \mathbf{N}(K)} d_x < \infty. \tag{6.14}$$

Let $\delta_0 > 0$ be as in (2.6) and let $\beta > 0$ be such that Proposition 6.11 holds for all $T > \beta \cdot \mathrm{H}_\iota(x)$. Let $\delta > 0$ be a constant given in the same proposition.

Applying Lemma 2.4 for $\mathbf{M}$ and $\mathbf{N}$ with κ therein, we have

$$\begin{aligned} &\sum_{x \in \mathbf{N}(K) : \mathrm{H}_\iota(x) > \beta^{-1} T} N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T) \\ &= \#\{xy \in \mathbf{N}(K)\mathbf{M}(K) : \mathrm{H}_\iota(x) > \beta^{-1} T, \ \ \mathrm{H}_\iota(xy) < T\} \\ &\le N_{\mathbf{M}}(\mathrm{H}_\iota, \kappa\beta) \cdot N_{\mathbf{N}}(\mathrm{H}_\iota, \kappa T \delta_0^{-1}) \\ &= O(T^{a_\iota - \epsilon}). \end{aligned} \tag{6.15}$$

Now applying Proposition 6.11, since $\sum_{x\in\mathbf{N}(K)} d_x < \infty$,

$$\sum_{x\in\mathbf{N}(K):\mathrm{H}_\iota(x)\le\beta^{-1}T} N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)$$
$$= \left(\sum_{x\in\mathbf{N}(K):\mathrm{H}_\iota(x)\le\beta^{-1}T} c_x\right) T^{a_\iota}(\log T)^{b_\iota-1} + O(T^{a_\iota}(\log T)^{b_\iota-1-\delta}).$$

Therefore as $T\to\infty$,

$$N(\mathrm{H}_\iota, T) = \sum_{x\in\mathbf{N}(K):\mathrm{H}_\iota(x)\le\beta^{-1}T} N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T) + O(T^{a_\iota-\epsilon})$$
$$= \left(\sum_{x\in\mathbf{N}(K):\mathrm{H}_\iota(x)\le\beta^{-1}T} c_x\right) T^{a_\iota}(\log T)^{b_\iota-1}(1+O((\log T)^{-\delta})).$$

Since $\sum_{x\in\mathbf{N}(K):\mathrm{H}_\iota(x)\le\beta^{-1}T} c_x = C(\mathrm{H}_\iota) + O(T^{-\epsilon})$, we have

$$N(\mathrm{H}_\iota, T) = C(\mathrm{H}_\iota)\cdot T^{a_\iota}(\log T)^{b_\iota-1}(1+O((\log T)^{-\delta}))$$

finishing the proof. □

We now construct the probability measure on X_ι in order to prove Theorem 1.6 in the introduction in a general case. We consider each $\tilde{\mu}_{x,\iota}$ as a measure on X_ι since $X_{\mathbf{M}}$ is a closed subspace of X_ι. For each $x\in\mathbf{N}(K)$, denote by $x.\tilde{\mu}_{x,\iota}$ the measure defined by

$$(x.\tilde{\mu}_{x,\iota})(\psi) := \tilde{\mu}_{x,\iota}(\psi_x)$$

where $\psi_x(g) = \psi(gx)$. Noting that each $x.\tilde{\mu}_{x,\iota}$ supported on $X_{\mathbf{M}}x$, defines a probability measure μ_ι on X_ι by

$$\mu_\iota = \sum_{x\in\mathbf{N}(K)} \frac{c_x}{c_{\mathrm{H}_\iota}}(x.\tilde{\mu}_{x,\iota}). \tag{6.16}$$

Note that $\mu_\iota = \tilde{\mu}_\iota$ in the case when ι is saturated (see Theorem 4.18 for the definition of $\tilde{\mu}_\iota$).

Theorem 6.17. *For any $\psi\in C(X_\iota)$,*

$$\lim_{T\to\infty}\frac{1}{N(\mathrm{H}_\iota, T)}\sum_{g\in\mathbf{G}(K):\mathrm{H}_\iota(g)<T}\psi(g) = \int_{X_\iota}\psi\, d\mu_\iota.$$

Proof. Let $\psi \in C(X_\iota)$. We write

$$\mu_T(\psi) = \frac{1}{N(\mathrm{H}_\iota, T)} \sum_{g\in\mathbf{G}(K):\, \mathrm{H}_\iota(g)<T} \psi(g),$$

$$\mu_{x,T}(\psi) = \frac{1}{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)} \sum_{g\in\mathbf{M}(K):\, \mathrm{H}_\iota(gx)<T} \psi(g) \quad \text{for each} \quad x \in \mathbf{N}(K).$$

Note that

$$\mu_T(\psi) = \sum_{x\in\mathbf{N}(K)} \frac{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)}{N(\mathrm{H}_\iota, T)} \mu_{x,T}(\psi). \tag{6.18}$$

Since $\{\mu_T : T \gg 1\}$ is a sequence of probability measures on a compact space X_ι, it suffices to prove that any convergent subsequence of μ_T, in the weak* topology, has the same limit μ_ι. Hence without loss of generality, we may assume that μ_T is convergent.

Let $\delta > 0$ be as in Proposition 6.11 and let $\beta > 0$ be such that the same proposition holds for all $T > \beta \cdot \mathrm{H}_\iota(x)$. By (6.15) and Theorem 6.2, there exists $\epsilon > 0$ such that

$$\sum_{x\in\mathbf{N}(K):\, \mathrm{H}_\iota(x)>\beta^{-1}T} \frac{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)}{N(\mathrm{H}_\iota, T)} \le O(T^{-\epsilon}).$$

Hence

$$\lim_{T\to\infty} \mu_T(\psi) = \lim_{T\to\infty} \sum_{x\in\mathbf{N}(K):\, \mathrm{H}_\iota(x)\le\beta^{-1}T} \frac{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)}{N(\mathrm{H}_\iota, T)} \mu_{x,T}(\psi). \tag{6.19}$$

For $x \in \mathbf{N}(K)$ such that $\mathrm{H}_\iota(x) \le \beta^{-1}T$, we deduce from Proposition 6.11 and Theorem 6.2 that

$$\frac{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)}{N(\mathrm{H}_\iota, T)} \le \frac{c_x}{c_{\mathrm{H}_\iota}} + \theta_x (\log T)^{-\delta},$$

where θ_x comes from the error terms in (6.3) and (6.12), and satisfies $\sum_{x\in\mathbf{N}(K)} \theta_x < \infty$. Since the sum in (6.19) is majorized by

$$\sum_{x\in\mathbf{N}(K)} \left(\frac{c_x}{c_{\mathrm{H}_\iota}} + \theta_x (\log T)^{-\delta} \right) < \infty,$$

we can apply the dominated convergence theorem to obtain

$$\lim_{T\to\infty} \mu_T(\psi) = \sum_{x\in\mathbf{N}(K)} \frac{c_x}{c_{\mathrm{H}_\iota}} \left(\lim_{T\to\infty} \mu_{x,T}(\psi) \right).$$

By Theorem 1.6, applied to $\mathbf{M}$ and the height function $g \mapsto \mathrm{H}_\iota(gx)$, we have for every $\psi \in C(X_\iota)$,

$$\lim_{T\to\infty} \mu_{x,T}(\psi) = \frac{1}{N_{\pi^{-1}(x)}(\mathrm{H}_\iota, T)} \sum_{g\in\mathbf{M}(K):\mathrm{H}_\iota(gx)<T} \psi_x(g) = \int_{X_\iota} \psi_x \, d\tilde{\mu}_{x,\iota}$$

where $\psi_x(g) = \psi(gx)$.

Therefore

$$\lim_{T\to\infty} \mu_T(\psi) = \sum_{x\in\mathbf{N}(K)} \frac{c_x}{c_{\mathrm{H}_\iota}} \cdot \tilde{\mu}_{x,\iota}(\psi_x).$$

□

7. Manin's and Peyre's conjectures

In this section, we now explain our main results in the context of Manin's conjecture on the asymptotic number of rational points of bounded height for Fano varieties. Let X be a smooth projective variety defined over K. For every line bundle class $[L]$ on X defined over K, there exists an associated height function $\mathrm{H}_\mathcal{L}$ on $X(K)$, unique up to the multiplication by bounded functions, via Weil's height machine (cf. [51, Theorem B. 3.2]). For example, if L is a very ample line bundle of X with a K-embedding $\psi_L : X \to \mathbb{P}^N$, then a height function $\mathrm{H}_\mathcal{L}$ on $X(K)$ is defined as

$$\mathrm{H}_\mathcal{L} := \mathrm{H} \circ \psi_L$$

for some height function H on $\mathbb{P}^N(K)$. We call a pair $\mathcal{L} = (L, \mathrm{H}_\mathcal{L})$ a metrized line bundle. Due to the freedom of choosing a height function H on $\mathbb{P}^N(K)$, $\mathrm{H}_\mathcal{L}$ is not uniquely determined and this is why we use the subscript $\mathcal{L}$ rather than L.

For a metrized ample line bundle $\mathcal{L} = (L, \mathrm{H}_\mathcal{L})$ on X and a subset U of X, set

$$N_U(\mathcal{L}, T) := \#\{g \in U \cap X(K) \ : \ \mathrm{H}_\mathcal{L}(g) \ < \ T\}.$$

The goal of Manin's conjecture is to obtain the asymptotic (as $T \to \infty$) of $N_U(\mathcal{L}, T)$ for some Zariski open subset U of X, possibly by passing to a finite extension field of K.

Two important geometric invariants here are:

$$a_L := \inf\{a \in \mathbb{Q}^+ : a[L] + [K_X] \in \Lambda_{\text{eff}}(X)\} \text{ — the Nevanlinna invariant of } L,$$

$$b_L := \text{ the codimension of the face of } \Lambda_{\text{eff}}(X) \text{ containing } a_L[L] + [K_X] \text{ in its interior}$$

where $[K_X]$ denotes the canonical line bundle class and $\Lambda_{\text{eff}}(X)$ denotes the cone of classes of effective line bundles on X.

Now let $\mathbf{G}$ be a connected semisimple adjoint algebraic group defined over K. Let X denote the projective K-variety, which is the wonderful compactification of $\mathbf{G}$ constructed by De Concini and Procesi [19] and by De Concini and Springer [20]. It is shown in [19] that X is a Fano variety.

One way of constructing X explicitly is to take the Zariski closure of the image of $\mathbf{G}$ in $\mathbb{P}(\mathrm{M}_N)$ under an irreducible faithful representation $\mathbf{G} \to \mathrm{GL}_N$ whose highest

weight is regular. A dominant weight χ is called regular if $\chi = \sum_{\alpha\in\Delta} m_\alpha\omega_\alpha$ with all $m_\alpha > 0$ where $\{\omega_\alpha : \alpha \in \Delta\}$ is the set of fundamental weights.

The Picard group $\mathrm{Pic}(X)_{\bar{K}}$ is isomorphic to the weight lattice of $\mathbf{G}$. Under this isomorphism, the simple roots α correspond to the boundary divisors D_α such that $X - \mathbf{G} = \cup_\alpha D_\alpha$, and the Galois action on $\mathrm{Pic}(X)_{\bar{K}}$ corresponds to the twisted Galois action (also called the $*$-action, see [56, 2.3]) on the weight lattice. Hence, the Picard group $\mathrm{Pic}(X)$ is freely generated by the line bundles corresponding to the orbits of the fundamental weights under the twisted Galois action. The closed cone $\Lambda_{\mathrm{eff}}(X)$ of the effective line bundles is the positive cone spanned by $D_{\Gamma_K.\alpha}$, $\alpha \in \Delta$, i.e.,

$$\Lambda_{\mathrm{eff}}(X) = \oplus\, \mathbb{R}_{\geq 0}\, [D_{\Gamma_K.\alpha}]$$

where the sum is taken over the Γ_K-orbits $\Gamma_K.\alpha$ in the set $\{\alpha \in \Delta\}$ of simple roots and $D_{\Gamma_K.\alpha} := \sum_{\beta\in\Gamma_K.\alpha} D_\beta$, and the anticanonical class $[-K_X]$ corresponds to $2\rho + \sum_{\alpha\in\Delta}\alpha$. Moreover any ample line bundle class $[L]$ of X over K corresponds to a regular dominant weight in such a way that if $[L'] := m[L]$ corresponds to $\chi \in X^*(\mathbf{T})$ for $m \in \mathbb{N}$, the restriction of $\mathrm{H}_{\mathcal{L}'}$ to $\mathbf{G}(K)$ coincides with a height function H_ι with respect to the irreducible representation ι defined over K with the highest weight λ_ι and $a_L = a_\iota$ and $b_L = b_\iota$ (cf. [50, Proposition 6.3]). In particular, $\mathrm{H}_{\mathcal{L}'}$ has a natural extension to $\mathbf{G}(\mathbb{A})$. We refer to [10, Ch.6] for a more detailed account on the wonderful compactification,and [54, 4.1] and [50, section 6] on metrized line bundles.

Therefore the following theorem, conjectued by Manin, follows from Theorem 1.2.

Theorem 7.1. *Let X be the wonderful compactification of a connected adjoint semisimple K-group $\mathbf{G}$, and $\mathcal{L} = (L, \mathrm{H}_{\mathcal{L}})$ a metrized ample line bundle on X. Then there exist $c_{\mathcal{L}} > 0$ and $\delta > 0$ such that*

$$N_{\mathbf{G}}(\mathcal{L}, T) = c_{\mathcal{L}} \cdot T^{a_L} (\log T)^{b_L - 1} (1 + O((\log T)^{-\delta})).$$

In order to describe the distribution of rational points, we construct a finite measure $\tau_{\mathcal{L}}$ on $X(\mathbb{A})$ first for each saturated ample line bundle $\mathcal{L}$ and then for any ample line bundle $\mathcal{L}$.

Definition 7.2. *We call an ample line bundle L on X saturated if the representation defined by the corresponding dominant weight is saturated.*

We note that if $\mathbf{G}$ is K-simple, every ample line bundle is saturated, and that the anticanonical line bundle $-K_X$ is always saturated for any $\mathbf{G}$. Batyrev and Tschinkel introduced the notion of a strongly saturated line bundle in [4]: A line bundle $\mathcal{L}$ is called *strongly saturated* if for any Zariski open dense subset U of X,

$$\lim_{T\to\infty} \frac{N_U(\mathcal{L}, T)}{N_{\mathbf{G}}(\mathcal{L}, T)} = 1. \tag{7.3}$$

Lemma 7.4. *A strongly saturated (see* (7.3)*) ample line bundle L is saturated.*

Proof. Suppose not. Then by Theorem 4.13, there exists a connected normal K-subgroup $\mathbf{M}$ of $\mathbf{G}$ such that $\iota|_{\mathbf{M}}$ is saturated and the volume of $B_T \cap \mathbf{M}(\mathbb{A})$ is of order $T^{a_L}(\log T)^{b_L-1}$. By Theorem 1.6, $\# B_T \cap \mathbf{M}(K)$ has the order of $T^{a_L}(\log T)^{b_L-1}$. This contradicts to the assumption that L is strongly saturated. $\square$

Lemma 7.5. *Let $\mathcal{L}$ and $\mathcal{L}'$ be metrizations of a saturated line bundle L and (B_T, W_f), and (B'_T, W'_f) be defined as above with respect to $\mathcal{L}$ and $\mathcal{L}'$ respectively. Then*

$$\lim_{T\to\infty} \frac{\tau_{W_f}(B_T \cap G_{W_f})}{\tau_{W'_f}(B'_T \cap G_{W'_f})} = \frac{\tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{\tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))}.$$

Proof. Let $V_f = W_f \cap W'_f$. By Proposition 4.16, it suffices to show that

$$\lim_{T\to\infty} \frac{\tau(B_T \cap G_{V_f})}{\tau(B'_T \cap G_{V_f})} = \frac{\tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{\tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))}. \tag{7.6}$$

Let S be a finite set such that $\mathrm{H}_{\mathcal{L}}$ and $\mathrm{H}_{\mathcal{L}'}$ are equal on $\mathbf{G}^S$. If we set $\mathrm{H}_{\mathcal{L},S} = \mathrm{H}_{\mathcal{L}}|_{\mathbf{G}_S}$, then it follows from (7.10) and Theorem 4.3 that

$$\frac{\tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{\tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))} = \frac{\int_{\mathbf{G}_S} \mathrm{H}_{\mathcal{L},S}(g)^{-a_L}\gamma_S(g)\, d\tau_S}{\int_{\mathbf{G}_S} \mathrm{H}_{\mathcal{L}',S}(g)^{-a_L}\gamma_S(g)\, d\tau_S}. \tag{7.7}$$

Theorem 4.13 with $S=\emptyset$ and (7.7) imply that the both sides of (7.6) stay the same when $\mathrm{H}_{\mathcal{L},S}$ and $\mathrm{H}_{\mathcal{L}',S}$ are replaced by constant multiples. Hence, we can assume that

$$\mathrm{H}_{\mathcal{L},S}(e) = \mathrm{H}_{\mathcal{L}',S}(e) = 1. \tag{7.8}$$

As in the proof of Lemma 4.27, we obtain

$$\begin{aligned}\tau(B_T \cap G_{V_f}) &= \int_{g\in\mathbf{G}_S} \tau^S(B_{T\,\mathrm{H}_{\mathcal{L},S}(g)^{-1}} \cap s_g G_{V_f} \cap \mathbf{G}^S)\, d\tau_S(g)\\ &\sim \left(\int_{g\in\mathbf{G}_S} \mathrm{H}_{\mathcal{L},S}(g)^{-a_L}\gamma_S(g) d\tau_S(g)\right)\cdot \tau^S(B_T \cap G_{V_f} \cap \mathbf{G}^S).\end{aligned}$$

Similarly,

$$\tau(B'_T \cap G_{V_f}) \sim \left(\int_{g\in\mathbf{G}_S} \mathrm{H}_{\mathcal{L}',S}(g)^{-a_L}\gamma_S(g) d\tau_S(g)\right)\cdot \tau^S(B'_T \cap G_{V_f} \cap \mathbf{G}^S).$$

Since by (7.8),

$$B_T \cap \mathbf{G}^S = B'_T \cap \mathbf{G}^S,$$

this finishes the proof. $\square$

Now we define a finite measure $\tau_{\mathcal{L}}$ on X which describes the asymptotic distribution of rational points with respect to a metrized ample line bundle $\mathcal{L} = (L, \mathrm{H}_{\mathcal{L}})$. If L is saturated, we set $\tau_{\mathcal{L}}$ to be $\tilde{\tau}_{\mathcal{L}}$ defined in the following Proposition. Let $W_{\mathcal{L}}$ denote the maximal compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ under which $\mathrm{H}_{\mathcal{L}}$ is bi-invariant.

Proposition 7.9. *For any metrized ample line bundle $\mathcal{L} = (L, \mathrm{H}_{\mathcal{L}})$, there exists a unique finite measure $\tilde{\tau}_{\mathcal{L}}$ on $X(\mathbb{A})$ such that for all $\psi \in C(X(\mathbb{A}))$ invariant under a co-finite subgroup of $W_{\mathcal{L}}$,*

$$\tilde{\tau}_{\mathcal{L}}(\psi) = d_K^{-\dim(X)/2} \cdot \sum_{\chi\in\Lambda} \left(\lim_{s\to a_L^+} (s-a_L)^{b_L} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_{\mathcal{L}}(g)^{-s} \chi(g)\, \psi(g)\, d\tau(g) \right) \tag{7.10}$$

Proof. Let ι be the representation with highest weight given by the regular dominant weight corresponding to L. Then $X(\mathbb{A}) = X_\iota$. By Theorem 4.18 it suffices to set

$$\tilde{\tau}_{\mathcal{L}} = d_K^{-\dim(X)/2} \cdot \gamma_{W_{\mathcal{L}}}(e) \cdot \tilde{\mu}_\iota.$$

□

For a general ample line bundle L, the variety X has an *asymptotic arithmetic $\mathcal{L}$-fibration* in the sense of [4]. By the results in section 6, there exist a connected semisimple K-group $\mathbf{N}$ and a surjective K-homomorphism $\pi : \mathbf{G} \to \mathbf{N}$ such that for each $x \in \mathbf{N}(K)$, there exist a finite measure $\tilde{\tau}_{x,\mathcal{L}}$ on $X(\mathbb{A})$ supported on $\pi^{-1}(x)(\mathbb{A})$ satisfying the following:

(1) for any $\psi \in C(X(\mathbb{A}))$ invariant under a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$,
$$\tilde{\tau}_{x,\mathcal{L}}(\psi) = d_K^{-\dim(X_M)/2} \cdot \sum_{\chi\in\Lambda_{\mathbf{M}}} \left(\lim_{s\to a_L^+} (s-a_L)^{b_L} \int_{\mathbf{M}(\mathbb{A})} \mathrm{H}_{\mathcal{L}}(gx)^{-s} \chi(g)\, \psi(gx)\, d\tau_{\mathbf{M}}(g) \right)$$
where $\mathbf{M} = \pi^{-1}(e)$, $\tau_{\mathbf{M}}$ is the Haar measure on $\mathbf{M}(\mathbb{A})$ with $\tau_{\mathbf{M}}(\mathbf{M}(K)\backslash\mathbf{M}(\mathbb{A})) = 1$, $\Lambda_{\mathbf{M}}$ is defined in the same way as Λ for $\mathbf{M}$ and X_M is the closure of $\mathbf{M}$ in X;

(2) there exists $c_x > 0$ such that as $T \to \infty$,
$$N_{\pi^{-1}(x)}(\mathcal{L}, T) \sim c_x \cdot T^{a_L} (\log T)^{b_L - 1}.$$

By Lemma 7.5, there exists $c_L > 0$ (independent of metrization) such that $c_x = c_L \cdot \tilde{\tau}_{x,\mathcal{L}}(X(\mathbb{A}))$ for each $x \in \mathbf{N}(K)$. Theorem 6.2 implies that

$$\sum_{x\in\mathbf{N}(K)} \tilde{\tau}_{x,\mathcal{L}}(X(\mathbb{A})) < \infty$$

and that the following defines a finite measure on X satisfying Theorem 7.11:

$$\tau_{\mathcal{L}} := \sum_{x\in\mathbf{N}(K)} \tau_{x,\mathcal{L}}.$$

Theorem 7.11. *For any metrized ample line bundle $\mathcal{L} = (L, \mathrm{H}_{\mathcal{L}})$ on X, and for any $\psi \in C(X(\mathbb{A}))$,*

$$\lim_{T\to\infty} \frac{1}{N_{\mathbf{G}}(\mathcal{L}, T)} \sum_{g\in\mathbf{G}(K):\mathrm{H}_{\mathcal{L}}(g)<T} \psi(g) = \frac{1}{\tau_{\mathcal{L}}(X(\mathbb{A}))} \int_{X(\mathbb{A})} \psi\, d\tau_{\mathcal{L}}.$$

Moreover, if L *is saturated and* $c_{\mathcal{L}}$ *is as in Theorem 7.1, the ratio* $\frac{c_{\mathcal{L}}}{\tau_{\mathcal{L}}(X(\mathbb{A}))}$ *is independent of the metrization* $\mathrm{H}_{\mathcal{L}}$*.*

Remark: Peyre [42] defined the Tamagawa measure $\tau_{-\mathcal{K}_X}$ on $X(\mathbb{A})$ associated with the anti-canonical metrized line bundle $-\mathcal{K}_X = (-K_X, \mathrm{H}_{-\mathcal{K}_X})$:

$$\tau_{-\mathcal{K}_X} := c_0 \cdot \lim_{s\to 1^+} (s-1)^{\mathrm{rank}(\mathrm{Pic}(X))} \left(\prod_{v\in R-S} L_v(s, \mathrm{Pic}(X)) \right) \cdot \mathrm{H}_{-\mathcal{K}_X}(g)^{-1} d\tau(g)$$

where $S \subset R$ is a finite subset of places with bad reduction, and $c_0 = d_K^{-\frac{\dim(X)}{2}} \cdot \prod_{v\in R-S} L_v(1, \mathrm{Pic}(X))^{-1}$ with d_K the discriminant of K.

Note that $a_{-K_X} = 1$, $b_{-K_X} = \mathrm{rank}(\mathrm{Pic}(X))$, and

$$\lim_{s\to 1^+} (s-1)^{\mathrm{rank}(\mathrm{Pic}(X))} \int_{\mathbf{G}(\mathbb{A})} \mathrm{H}_{-\mathcal{K}_X}(g)^{-s} \chi(g) d\tau(g) = 0$$

for all $\chi \neq 1$. Hence for $\mathcal{L} = -\mathcal{K}_X$, (7.10) gives Peyre's measure $\tau_{-\mathcal{K}_X}$. An analog of Peyre's measure for general line bundles was also introduced in [4], but the measure $\tau_{\mathcal{L}}$ seems to be different, in general, from the measure defined in [4].

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